(Sets) مجموعات

1

1. The numbers 0, 1, 2, 3, 4, 5, ---- are called Natural numbers which is denoted by IN.
$IN = \{0, 1, 2, 3, 4, 5, ----\}$

2. The numbers ----, -3, -2, -1, 0, 1, 2, 3, ----
are called integer numbers and the set of all integer numbers is denoted by $\mathbb{Z}$
Observe That $IN \subseteq \mathbb{Z}$

3. The numbers on the form $\frac{p}{q}$ where p, q are integers and $q \neq 0$ are called rational numbers, the set of all rational numbers is denoted by $\mathbb{Q}$.
Note $\mathbb{Z} \subseteq \mathbb{Q}$
because every number a in $\mathbb{Z}$ can be written as $\frac{a}{1}$

4. numbers that can not be written as the ratio of two integers ~~is~~ are called irrational numbers
The set of all irrational numbers is denoted by $\mathbb{Q}'$
equivalent definition to irrational number
have infinitely non repeating decimal places such as
$\sqrt{2} = 1.414213562 ---$ and $\pi = 3.1415926 5 ----$

Remark : $\frac{22}{7}$ approxiamation value to π

5. Rational numbers together with irrational numbers are called Real numbers. The set of all Real numbers

5. Real number line :- Real numbers can be re[presented] by a points on line called Real number line.

6. The numbers on the form $a+ib$, where $a,b \in \mathbb{R}$ are called complex numbers. The set of all complex numbers is denoted by $\mathbb{C}$

Remark: every real number is complex number because $a \in \mathbb{R}$ can be written as $a+i0$

7 Set : is the collection of objects (elements or numbers)

8 : The following nine types of subsets of $\mathbb{R}$ are called intervals

$(a,b) = \{x \in \mathbb{R} : a < x < b\}$

$(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$.

$[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$

$(-\infty,b) = \{x \in \mathbb{R} : -\infty < x < b\} = \{x \in \mathbb{R} : x < b\}$.

$(-\infty,b] = \{x \in \mathbb{R} : x \leq b\}$.

$[a,\infty) = \{x \in \mathbb{R} : x \geq a\}$

$(a,\infty) = \{x \in \mathbb{R} : x > a\}$.

$(-\infty,\infty) = \mathbb{R}$.

Theorem (properties of Inequalities).

(1) if $a<b$ and $b<c$, Then $a<c$.

(2) if $a<b$, Then $a-c<b-c$, and $a+c<b+c$.

(3) if $a<b$, Then $ac<cb$ for $c>0$
and $ac>cb$ for $c<0$

$a<b$ and $c<d$, then $a+c<b+d$.

If a and b are both positive or both negative and $a<b$, then $\frac{1}{a} > \frac{1}{b}$

Example solve $3+7x \leq 2x-9$

$$\Rightarrow 3+7x-2x \leq 2x-9-2x \quad (\text{by } 2)$$

$$\Rightarrow 3+5x \leq -9$$

$$\Rightarrow -3+3+5x \leq -9+(-3) \quad (\text{by } 2)$$

$$\Rightarrow 5x \leq -12.$$

$$\Rightarrow \frac{1}{5}(5x) \leq \frac{1}{5}(-12) \quad (\text{by } (3))$$

$$\Rightarrow x \leq \frac{-12}{5}.$$

Example solve $7 \leq 2-5x \leq 9$

$$\Rightarrow -2+7 \leq -2+2-5x \leq -2+9$$

$$\Rightarrow 5 \leq -5x \leq 7$$

$$\Rightarrow -\frac{1}{5}(5) \geq x \geq -\frac{7}{5}$$

$$\Rightarrow -1 \geq x \geq -\frac{7}{5}$$

$$\Rightarrow -\frac{7}{5} \leq x \leq -1$$

Example solve $x^2-3x>10$

$$\Rightarrow x^2-3x-10>0$$

$$\Rightarrow (x-5)(x+2)>0$$

then either $x+2=0 \Rightarrow x=-2$

$x-5=0 \Rightarrow x=5$

$\leq b$ and $b \leq a$. What can you say a and b? 3

H.W 3 list the elements in the set.

(a) $\{x : x^2 - 5x = 0\} = \{x : x(x-5) = 0\}$
$= \{0, 5\}$.

(b) $\{x : x$ is an integer satisfying $-2 < x < 3\}$
$= \{-1, 0, 1, 2\}$.

H.W 4 express the following sets in the notation $\{---\}$ depeding (depending on variables)

$\mathbb{Z}_o = \{1, 3, 7, 9, ---\}$
$= \{n : 2n+1,\ n \in \{0, 1, 2, ---\}\}$.

$\mathbb{Z}_e = \{0, 2, 4, 6, 8, ---\}$
$= \{2n : n \in \{0, 1, 2, ---\}\}$.

H.W 5 list all subsets of

(a) $\{a_1, a_2, a_3\}$ (b) ϕ.

H.W 6 sketch the set on a coordinate line

(1) $[-3, 2] \cup [1, 4]$

(2) $(-2, 4) \cap (0, 5]$

(3) $[4, 6] \cup [8, 11]$.

$$- - - \quad 0 \quad + + + \quad 5/2 \quad - - -$$

$$\{x \in \mathbb{R} : x < 0\} \cup \{x \in \mathbb{R} : x > \tfrac{5}{2}\}$$

$$= (-\infty, 0) \cup (\tfrac{5}{2}, \infty).$$

(4) prove or disprove if $\sqrt{a^2} < \sqrt{b^2}$ Then $a < b$.

In general the statement is not true

take $a = 1$; $b = -5$

$\sqrt{(1)^2} < \sqrt{(-5)^2}$ But $1 \not< -5$

H.W (1) 2023

$$(x-1)(x-2)(x-3)(x-4) < 0$$

H.W (2)

$$(x-1)^2 (x+4) < 0$$

(5) $\frac{2}{x} < 3$

case 1: if $x > 0$.

Then $2 < 3x \Rightarrow x > \frac{2}{3}$

$$\{x \in \mathbb{R} : x > \tfrac{2}{3}\} = (\tfrac{2}{3}, \infty).$$

case 2: if $x < 0$.

$2 > 3x \Rightarrow x < \frac{2}{3}$

$$\{x \in \mathbb{R} : x < 0\}$$

Inequality

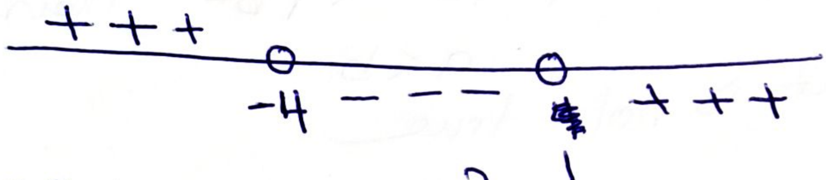
مراجعة

① $(x-1)(x+4)<0$

To find the main points.

$(x-1)(x+4)=0$

either $x=1$ or $x=4$



$\{x\in R: -4<x<1\} = (-4,1)$

② $x^2-6x+5>0$

$x^2-6x+5=0$

$\Rightarrow (x-5)(x-1)=0$

$\Rightarrow$ either $x=1$ or $x=5$

$\{x\in R: x>5\} \cup \{x\in R: x<1\}$

$= (5,\infty) \cup (-\infty,1)$

③ $5x-2x^2>0$

To find the main points:

$5x-2x^2=0$

$\Rightarrow x(5-2x)=0$ either $x=0$ or $x=\frac{5}{2}$

$3 \leq 4-2x < 7$

$-4+3 \leq -2x < -4+7$

$\left(-1 \leq -2x < 3\right)$ Multiply $\left(-\frac{1}{2}\right)$

$\frac{1}{2} \geq x > \frac{-3}{2}$.

Hence $\frac{-3}{2} < x < \frac{1}{2}$.

(5) $x^2 > 9 \Rightarrow x^2 - 9 > 0$

$\Rightarrow (x-3)(x+3) > 0$

Then either $x = 3$ or $x = -3$

So, we may have two intervals $(-\infty, -3)$ and $(3, \infty)$. $(-3, 3)$.

	Test number	
$(-\infty, -3)$	-4	$(-)(-) = +$
$(-3, 3)$	0	$(-)(+) = -$
$(3, \infty)$	4	$(+)(+) = +$

Hence, the solution set is $(-\infty, -3) \cup (3, \infty)$.

H.W 7 solve the following

$\frac{\frac{1}{2}x - 3}{4+x} > 1$; $\frac{1}{x+1} \geq \frac{3}{x-2}$

$x^3 - 3x + 2 \leq 0$

Definition: The absolute of a real number is
by $|a|$ is defined by $|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$

Example ①

$|5| = 5$, $\left|-\frac{4}{7}\right| = -\left(-\frac{4}{7}\right) = \frac{4}{7}$

Example ② solve $|x-3| = 4$

$\Rightarrow (x-3) = 4 \Rightarrow x = 7$

$-(x-3) = 4 \Rightarrow x - 3 = -4$

$\Rightarrow x = -4 + 3 = -1.$

Example ③ solve $|3x-2| = |5x+4|$.

Then either $(3x-2) = (5x+4)$ or

$(3x-2) = -(5x+4)$

$\Rightarrow (3x-2) = (5x+4).$

$\Rightarrow -2 = 5x - 3x + 4$

$\Rightarrow -2 - 4 = 5x - 3x$

$\Rightarrow -6 = 2x \Rightarrow x = -3$

or $3x - 2 = -5x - 4$

$\Rightarrow 3x + 5x = -4 + 2$

$\Rightarrow 8x = -2 \Rightarrow x = \frac{-2}{8} = -\frac{1}{4}$

E/ if A and B are points on the coordinate line with coordinates a and b, then the distance d between A and B is $d = |b - a|$

Remark

$|x-a|$ is the distance between x and a.

$|x+a| = |x-(-a)|$ is the distance between x and $(-a)$.

$|x| = |x-0|$ is the distance between x and 0

Remark

for $K > 0$

1) $|x-a| < K \iff -K < x-a < K$

$\iff a-K < x < a+K$

2) $|x-a| > K \iff x-a < -K$ or $x-a > K$

$\iff x < a-K$ or $x > a+K$

Example solve the following

(a) $|x-3| < 4$

$\Rightarrow -4 < x-3 < 4$

$\Rightarrow -4+3 < x < 4+3$

$\Rightarrow -1 < x < 7$

(b) $|x+4| \geq 2$

Then

$x+4 \leq -2$ or $x+4 \geq 2$

$x \leq -2-4$ or $x \geq 2-4$

$x \leq -6$ or $x \geq -2$

Then the solution.

$(-\infty, -6] \cup [-2, \infty)$.

E/ if A and B are points on the coordinate line. with coordinates a and b, Then the distance d between A and B is $d = |b-a|$

Remark

$|x-a|$ is the distance between x and a.

$|x+a| = |x-(-a)|$ is the distance between x and $(-a)$.

$|x| = |x-0|$ is the distance between x and 0

Remark for $k > 0$

1) $|x-a| < k \iff -k < x-a < k$
$\iff a-k < x < a+k$

2) $|x-a| > k \iff x-a < -k$ or $x-a > k$
$\iff x < a-k$ or $x > a+k$

Example solve the following

(a) $|x-3| < 4$

$\Rightarrow -4 < x-3 < 4$

$\Rightarrow -4+3 < x < 4+3$

$\Rightarrow -1 < x < 7$

(b) $|x+4| \geq 2$

Then

$x+4 \leq -2$ or $x+4 \geq 2$.

$x \leq -2-4$ or $x \geq 2-4$

$x \leq -6$ or $x \geq -2$

Then the solution.

$(-\infty, -6] \cup [-2, \infty)$.

Solve This example

$$\left|x+\frac{1}{x}\right| > 2$$

$$\Rightarrow \left|\frac{x^2+1}{x}\right| > 2$$

$$\Rightarrow \frac{x^2+1}{|x|} > 2$$

Case ①: $x \geq 0$, Then

$$x^2+1 > 2x$$

$$x^2+2x+1 > 0$$

$$(x-1)(x-1) > 0$$

$$\begin{array}{ccc} & 1 & \\ +++ & ① & +++ \end{array}$$

$$S_1 = \mathbb{R} \setminus \{1\}$$

Case ②: $x < 0$, Then

$$x^2+1 < 2x$$

$$x^2-2x+1 < 0$$

$$(x-1)^2 < 0$$

there is no solution.

Example

$$4-x \geq |5x+1|$$

Case 1

if $5x+1 \geq 0$

then $5x \geq -1$

$$x \geq \frac{-1}{5}$$

$$5x+1 \leq 4-x$$

ram For any real number a (5)

$\sqrt{a^2} = |a|$

Since $\sqrt{a^2} = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases} = |a|$

For example

$\sqrt{9}$ is coming from $\sqrt{(3)(3)}$ or $\sqrt{(-3)(-3)}$

$\sqrt{(3)(3)} = 3$, $\sqrt{(-3)(-3)} = 3$

Theorem if a and b are real numbers

Then $|-a| = |a|$

$|ab| = |a||b|$

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.

proof: ① $|-a| = \sqrt{(-a)^2} = \sqrt{a^2} = |a|$

② $|ab| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2} . \sqrt{b^2} = |a||b|$

③ $\left|\frac{a}{b}\right| = \sqrt{\left(\frac{a}{b}\right)^2} = \sqrt{\frac{a^2}{b^2}} = \frac{\sqrt{a^2}}{\sqrt{b^2}} = \frac{|a|}{|b|}$.

proposition: For any a_i is real number

1) $|a_1 . a_2 ---- a_n| = |a_1| |a_2| ---- |a_n|$.

2) $|a^n| = (|a|)^n$

proof: H.W 8

Example: solve the following

$$\frac{1}{|2x-3|} > 5$$

it follows $|2x-3| < \frac{1}{5}$

$$-\frac{1}{5} < 2x-3 < \frac{1}{5}$$

$$\Rightarrow -\frac{1}{5}+3 < 2x < \frac{1}{5}+3$$

$$\Rightarrow \frac{14}{5} < 2x < \frac{16}{5} \Rightarrow \frac{7}{5} < x < \frac{8}{5}$$

Theorem

if a and b are real numbers, Then $|a+b| \leq |a| + |b|$

proof: observe That for all $a \in \mathbb{R}$, $a \leq |a|$ — ①

To verify This we have to check two cases

if $a \geq 0$, Then $a \leq |a| = a$

if $a < 0$, Then $a < 0 \leq |a|$

By applying $(-a)$ in ①, we had

$\Rightarrow -a \leq |-a| = |a|$ ---- ②

if $a+b \geq 0$, then $|a+b| = a+b \leq |a|+|b|$

if $a+b<0$, then $|a+b| = -(a+b)$
$= -a-b \leq |a|+|b|$

Hence $|a+b| \leq |a|+|b|$

Theorem

$$\big||a|-|b|\big| \leq |a-b|, \text{ for every } a,b \in \mathbb{R}$$

proof: since $|b| = |a+b-a| \leq |a| + |b-a|$
and since $|a| = |b+a-b| \leq |b| + |a-b|$

Then $|b|-|a| \leq |b-a| = |a-b|$
and $|a|-|b| \leq |a-b|$

But $$\big||a|-|b|\big| = \begin{cases} |a|-|b| & \text{if } |a| \geq |b| \\ |b|-|a| & \text{if } |a|<|b| \end{cases}$$

Therefore $\big||a|-|b|\big| \leq |a-b|$.

Prove That $|a-b| \leq |a| + |b|$.

$$|a-b| = |a+(-b)| \leq |a| + |-b|$$
$$= |a| + |b|$$

...sian coordinate system consists of two ...rpendicular coordinate lines called coordinate axis

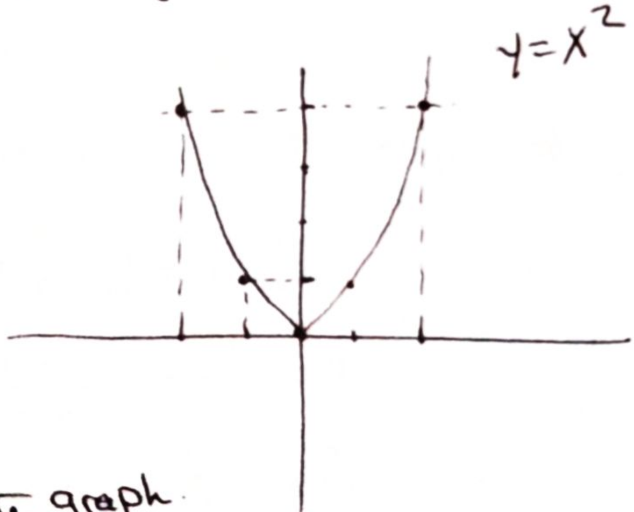
- the intersection of the axis is called the origin of the coordinate system.

- the horizontal line is called the x-axis and the vertical line is called y-axis.

Definition: the set of all solutions of an equation in x and y is called the solution set of the equation. and the set of all points in the xy-plane whose coordinate are members of the solution set is called the graph of function.

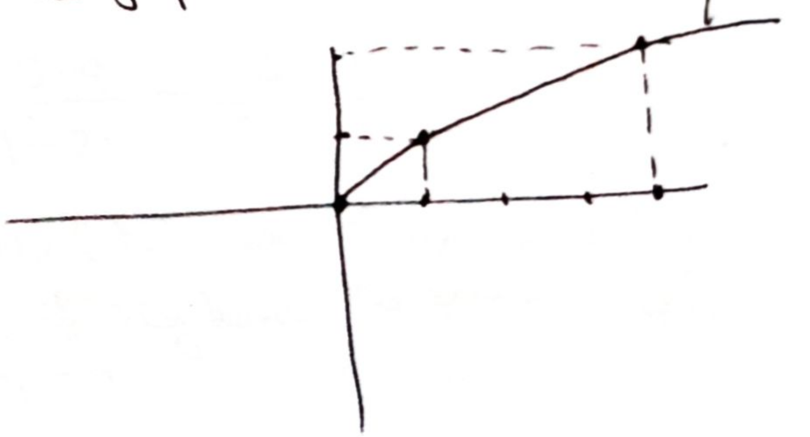
Example 1: sketch the graph

x	$y=x^2$
-2	4
-1	1
0	0
1	1
2	4



Example sketch the graph

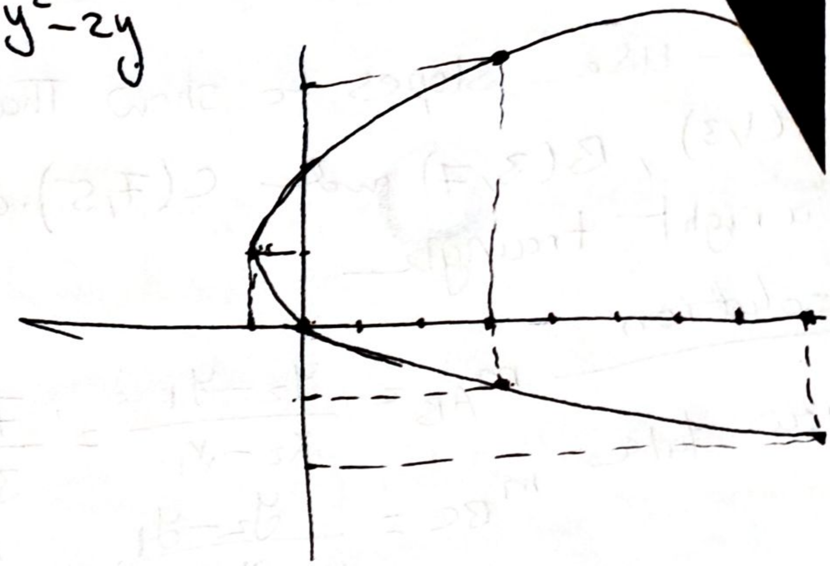
x	$y=\sqrt{x}$
0	0
1	1
4	2



Example: sketch the graph. $y^2-2y-x=0$

solution: $x=y^2-2y$

y	$x=y^2-2y$
0	0
-1	3
-2	8
1	-1
2	0
3	3



(3, -1)
(8, -2).
(-1, 1).
(0, 2).

Definition if. $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are point on the non vertical line, then the slope m is defined by

$$m=\frac{y_2-y_1}{x_2-x_1}$$

Example: In each part find the slope of the line through

(a) the points. $\overset{x_1\,y_1}{(6,2)}$ and $\overset{x_2\,y_2}{(9,8)}$

$$m=\frac{8-2}{9-6}=\frac{6}{3}=2$$

(b) The points $(2,9)$ and $(4,3)$.

Then

$$m=\frac{3-9}{4-2}=\frac{-6}{2}=-3$$

Theorems

(1) two non vertical lines with slopes m_1 and m_2 are parallel. iff they have the same slope. That is $m_1=m_2$.

Slopes

Q1 Use slopes to show That The points $A(1,3)$, $B(3,7)$ and $C(7,5)$ are vertices of a right traingle

solution

$$m_{AB} = \frac{y_2-y_1}{x_2-x_1} = \frac{7-3}{3-1} = \frac{4}{2} = 2$$

Now take

$$m_{BC} = \frac{y_2-y_1}{x_2-x_1} = \frac{5-7}{7-3} = \frac{-2}{4} = \frac{-1}{2}$$

since $m_{AB} \cdot m_{BC} = 2\left(\frac{-1}{2}\right) = -1$

Therefore AB and BC are perpendicular

Q2 (H-W) use slopes to determine whether the given points lie on the same line.

$P_1(1,1)$, $P_2(-2,-5)$, $P_3(0,-1)$.

Q3 (H.W) Find x if the slope of the line Through $(1,2)$ and $(x,0)$ is The negative of The slope of The line Through $(4,5)$ and $(x,0)$.

Q4 given That a point $(k,4)$ is on the line Through $(1,5)$ and $(2,-3)$, find K?

solution

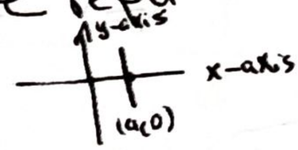
...wo non vertical lines with slopes m_1 and m_2 are perpendicular iff $m_1 \cdot m_2 = -1$

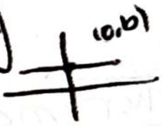
$$m_1 = \frac{-1}{m_2} \quad \text{or} \quad m_2 = \frac{-1}{m_1}.$$

3. The line passing Through $P_1(x_1, y_1)$ and having slope m is given by the equation

$$y - y_1 = m(x - x_1).$$

4. the vertical line Through $(a, 0)$ and the horizontal line Through $(0, b)$ are repersented by the equation $x = a$ and $y = b$.





5. The line with y-intercept b and slope m is given by the equation $y = mx + b$

Example.

Equation	slope	y-intercept
$y = 3x + 7$	$m = 3$	$b = 7$
$y = -x + 1/2$	$m = -1$	$b = 1/2$
$y = 2$	$m = 0$	$b = 2$.

Example Find the slope-intercept form of the equation $(y = mx + b)$.

(a) slope is -9, crosses the y-axis at $(0, 4)$

$m = -9$. / $\overline{(\text{y-intercept}) = 4}$.

$$y = -9x + 4.$$

solution

$$m = \frac{-3-5}{2-1} = \frac{-8}{1} = -8$$

$\Leftrightarrow y - y_1 = m(x - x_1)$

$\Leftrightarrow y - 5 = -8(x-1).$

$\Leftrightarrow y - 5 = -8x + 8$

$\Leftrightarrow y = -8x + 13$

$\Leftrightarrow 4 = -8K + 13 \Leftrightarrow 4 - 13 = -8K$

$\Leftrightarrow -9 = -8K \Leftrightarrow K = \frac{9}{8}$

another method

$(K,4)$; $(1,5)$; $(2,-3)$

$m_1 = \frac{5-4}{1-K}$; $m_2 = \frac{-3-5}{2-1}.$

$\Leftrightarrow \frac{1}{1-K} = \frac{-8}{1} \Leftrightarrow -8(1-K) = 1 \Leftrightarrow K = \frac{9}{8}$

, and passes through ori[gin]
then $m=1$, since passes through $(0,0)$
then $b=0$.

$y = mx + b$

$\Rightarrow y = 1 \cdot x + 0 \Rightarrow y = x$.

(c) passes through $(5,-1)$ and perpendicular to $y=3x+4$

then $m_1 = 3$; so $m_2 = \frac{-1}{m_1} = \frac{-1}{3}$

and since

$$y - y_1 = m(x - x_1)$$
$$y+1 = -\frac{1}{3}(x-5)$$
$$y+1 = -\frac{1}{3}x + \frac{5}{3}$$
$$y = -\frac{1}{3}x + \frac{5}{3} - \frac{3}{3}$$
$$y = -\frac{1}{3}x + \frac{2}{3}$$

(d) passes through points $(3,4)$ and $(+2,-5)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5-4}{2-3} = \frac{-9}{-1} = 9$$

$\Rightarrow (y - y_1) = m(x - x_1)$

$\Rightarrow y - 4 = 9(x-3)$

$\Rightarrow y - 4 = 9x - 27$

$\Rightarrow y = 9x - 27 + 4$
$= 9x - 23$

Distance and Circle equations

Thm: The distance d between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in coordinate plane is given by

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

Example Find the distance between the points $(-2,3)$ and $(1,7)$

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

$$= \sqrt{(\underset{1+2}{-2-1})^2 + (7-3)^2} = \sqrt{9+16} = \sqrt{25}$$
$$= 5$$

theorem (Midpoint Formula.) ; the midpoint of the line segment joining two points (x_1, y_1) and (x_2, y_2) in a coordinate plane $\left(\frac{1}{2}(x+x_2), \frac{1}{2}(y_1+y_2)\right)$.

Example Find the midpoint of the line segment joining $(3,-4)$ and $(7,2)$.

$$\text{midpoint} = \left(\frac{1}{2}(x_1+x_2), \frac{1}{2}(y_1+y_2)\right)$$
$$= \left(\frac{1}{2}(3+7), \frac{1}{2}(-4+2)\right) = (5,-1).$$

Circle :- if (x_0, y_0) is a fixed point in the plane, then the circle ~~radius~~ r centred at (x_0, y_0) is the set of all points in the plane whose distance from (x_0, y_0) is r, Thus a point (x, y) lie on this circle iff

$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

$$r^2 = (x-x_0)^2 + (y-y_0)^2$$

which is called standard form of the equation of circle

...ide by (2), Then we get

$$x^2+y^2+12x-\frac{81}{2}=0$$

$$\Rightarrow x^2+12x+y^2-\frac{81}{2}=0$$

$$\Rightarrow x^2+12x+36-36+y^2-\frac{81}{2}=0$$

$$\Rightarrow (x+6)^2+(y-0)^2-\frac{153}{2}=0$$

$$\Rightarrow (x+6)^2+(y-0)^2=\frac{153}{2}$$

$c(-6,0)$ and $r=\sqrt{\frac{153}{2}}$.

H.W Find x and y if $(4,-5)$ is the midpoint of the line segment joining $(-3,2)$ and (x,y).

H.W prove that $(0,-2)$, $(-4,8)$ and $(3,1)$ lie on the circle with center $(-2,3)$.

H.W Find the standard equation of the circle statisfying the given condition:
A diameter has end points $(6,1)$ and $(-2,3)$.

Functions

Definition: A function is relation between two sets A and B, such for each $x \in A$, There is unique $y \in B$ defined by $f(x) = y$

Definition: if $y = f(x)$, Then The set of all possible inputs is called the domain of. (denoted by D) and the set of all possible outcome is called range (results) (which is denoted by R).

Example: Find the domain and range for the following

① $y = x^2$ $\quad D = \mathbb{R} \quad R = \mathbb{R}^+ \cup \{0\}$.

② $y = x$ $\quad D = \mathbb{R} \quad R = \mathbb{R} = (-\infty, \infty)$

③ $f(x) = 2 + \sqrt{x-1}$; $D = \{x \in \mathbb{R} : x - 1 \geq 0\}$

$= \{x \in \mathbb{R} : x \geq 1\}$

$= [1, \infty)$.

$R = \{y \in \mathbb{R} : y \geq 2\}$

$= [2, \infty)$.

④ $y = f(x) = \dfrac{x+1}{x-1}$; Then $D = \{x \in \mathbb{R} : x - 1 \neq 0\}$

$= \{x \in \mathbb{R} : x \neq 1\}$

$= (-\infty, 1) \cup (1, \infty)$

$y = \dfrac{x+1}{x-1}$

To find $x = \phi(y)$

$\Rightarrow x + 1 = y(x - 1)$.

$\Rightarrow x + 1 = yx - y$

$\Rightarrow x - yx = -1 - y$

$x(1 - y) = -1 - y$

$x = \dfrac{-(1+y)}{1-y} = \dfrac{y+1}{y-1}$

$R = \{y \in \mathbb{R} : y - 1 \neq 0\}$

$R = (-\infty, 1) \cup (1, \infty)$.

Definition: Given a function f and g, we def[ine]

$(f+g)(x) = f(x) + g(x)$. $\quad (f/g)(x) = \frac{f(x)}{g(x)}$.

$(f-g)(x) = f(x) - g(x)$.

$(f \cdot g)(x) = f(x) \cdot g(x)$.

<u>Example</u> let $f(x) = 1+\sqrt{x-2}$ and $g(x) = x-3$

Then $(f+g)(x) = f(x) + g(x)$
$= 1+\sqrt{x-2} + (x-3)$.

$D_{f+g} = \{x \in \mathbb{R} : x-2 \geq 0\}$.

$(f-g)(x) = f(x) - g(x)$
$= 1+\sqrt{x-2} - (x-3) = 1+\sqrt{x-2} - x + 3$

$D_{f-g} = \{x \in \mathbb{R} : x-2 \geq 0\}$
$= \{x \in \mathbb{R} : x \geq 2\}$.

$(f \cdot g)(x) = (1+\sqrt{x-2})(x-3)$

$D_{f \cdot g} = \{x \in : x \geq 2\} = [2, \infty)$.

Definition: Given functions f and g, then the composition of f with g denoted by

$(f \circ g)(x) = f(g(x))$.

Ex Let $f(x) = x^2+3$, $g(x) = \sqrt{x}$ Find $f \circ g$, $g \circ f$ and $f \circ f$.

① $f \circ g(x) = f(\sqrt{x}) = (\sqrt{x})^2 + 3 = x+3$

② $g \circ f(x) = g(x^2+3) = \sqrt{x^2+3}$ $\quad D = (-\infty, \infty)$
$= \mathbb{R}$.

$\circ f(x) = f(x^2+3) = (x^2+3)^2 + 3$

$= x^4 + 2(3)x^2 + 9 + 3$

$= x^4 + 6x^2 + 12$

$fog(3) = (\sqrt{3})^2 + 3 = 3+3 = 6$

Remark $fogoh(x) = fog(h(x)) = f(g(h(x)))$.

Example: Find $(fogoh)(x)$ if $f(x) = \sqrt{x}$; $g(x) = 1/x$

$h(x) = x^3$

Then $fogoh(x) = fog(x^3)$

$= f(g(x^3)) = f\left(\frac{1}{x^3}\right) = \sqrt{\frac{1}{x^3}}$.

Example Express $h(x) = (x-4)^5$ as a composition of two functions $f(x) = x-4$, $g(x) = x^5$

$gof(x) = g(x-4) = (x-4)^5$

Example: Express $h(x) = \sin x^3$ as composition of two functions $f(x) = \sin x$ $g(x) = x^3$.

$fog(x) = f(x^3) = \sin x^3$

Definition: A function f is called even if $f(x) = f(-x)$

Ex/ $f(x) = x^2$

$\Rightarrow f(-x) = (-x)^2 = f(x)$.

$f(x) = 3$

$\Rightarrow f(-x) = 3 = f(x)$.

Definition: A function f is called odd if

$$f(-x) = -f(x).$$

Ex/ $f(x) = x^3$

$$\Rightarrow f(-x) = (-x)^3 = -x^3 = -f(x).$$

Hence f is an odd.

- there are many functions that are non-odd and non-even functions.

Let $f(x) = x+2$

$$f(-x) = -x+2 \neq f(x)$$

$$-f(x) = -x-2 \neq f(-x).$$

Hence f is not even, and its not odd.

Definition: A function $f: A \to B$ is called one to one if $\forall x_1, x_2 \in A$, with $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

with $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

prove $f: \underset{\mathbb{R}}{A} \to \underset{\mathbb{R}}{B}$ defined by $f(x) = 2x+3$ is one to one

Let $x_1, x_2 \in \mathbb{R}$, s.t. $f(x_1) = f(x_2)$

$$\Rightarrow 2x_1 + 3 = 2x_2 + 3$$

$$\Rightarrow 2x_1 = 2x_2$$

$$\Rightarrow x_1 = x_2.$$

Hence f is 1-1.

Determine wether $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is one to one or not

nition: a function $f:A \to B$ is onto if $f(A)=B$
the other words, $\forall\, y \in B$, there is $x \in A$ s.t.
$$f(x)=y.$$

prove that $f:\mathbb{R} \to \mathbb{R}$ is onto.
$f(x)=2x+3$

$$\Rightarrow \quad y=2x+3.$$
$$y-3=2x \Rightarrow x=\frac{y-3}{2}.$$

$$f(x)=f\left(\frac{y-3}{2}\right)=2\left(\frac{y-3}{2}\right)+3=y-3+3=y$$

Definition: (9:) A function $f:A \to B$ is called bijective if it is one to one and onto.

Example: $f:\mathbb{R} \to \mathbb{R}$ defined by $f(x)=\frac{x}{2}$.

Definition: A function $f:A \to B$ have an inverse such that $f^{-1}:B \to A$ if f is 1-1, and onto.

Example let $f:\mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$
$f(x)=x+1$. then f is 1-1, onto.

$$f^{-1}:\mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$$
$f^{-1}(y)=y-1$. Domain $=\mathbb{R}^+ \cup \{0\}$
$\Rightarrow f^{-1}(x)=x-1$ Range $=\mathbb{R}^+ \cup \{0\}$.

Definition :- if the values of $f(x)$ can be made as close as we like to L by taking value of x sufficiently close to a, then we write.

$$\lim_{x \to a} f(x)=L$$

② one side limits: if the values of $f(x)$ can made as close we like to L by taking values of x sufficiently close to a (but greater than a)

$$\lim_{x \to a^+} f(x) = L.$$

which is read "the limit of $f(x)$ as x approaches a from the right is L"

Similarly; if the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a).

$$\lim_{x \to a^-} f(x) = L.$$

which is read "the limit of $f(x)$ as x approaches a from the left is L".

③ two sided limit: the two sided limit of functions $f(x)$ exists at a if and only if the one side limits exist at a and have the same value.

that is.

$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x).$$

Theorem let a and k be a real numbers

$$\lim_{x \to a} k = k, \quad \lim_{x \to a} x = a, \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

...ple 5

$\lim\limits_{x\to 2} 4 = 4$, $\lim\limits_{x\to 2} x = 2$, $\lim\limits_{x\to 0^-} \frac{1}{x+2} = -\infty$

$\lim\limits_{x\to 0^+} \frac{1}{2x} = +\infty$

<u>Theorem</u> Let a be a real number, and suppose that $\lim\limits_{x\to a} f(x) = L_1$ and $\lim\limits_{x\to a} g(x) = L_2$, Then

(1) $\lim\limits_{x\to a} [f(x) \mp g(x)] = \lim\limits_{x\to a} f(x) \mp \lim\limits_{x\to a} g(x)$

$= L_1 \mp L_2$

(2) $\lim\limits_{x\to a} [f(x) \cdot g(x)] = \lim\limits_{x\to a} f(x) \lim\limits_{x\to a} g(x)$

$= L_1 \cdot L_2$

(3) $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \frac{L_1}{L_2}$

(4) $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)} = \sqrt{L_1}$ for $L_1 > 0$ if n is even.

(5) $\lim\limits_{x\to a} x^n = (\lim\limits_{x\to a} x)^n = a^n$

(6) $\lim\limits_{x\to a} (k \cdot f(x)) = \lim\limits_{x\to a} k \cdot \lim\limits_{x\to a} f(x)$

$= k \cdot L_1$

Example:- Find $\lim\limits_{x\to -1} x^2 - x + 1 = (-1)^2 - 1 + 1$

$= 1$

Example: Find $\lim_{x \to 2} \frac{5x^3+4}{x-3}$

$$= \frac{\lim_{x \to 2} 5x^3+4}{\lim_{x \to 2} x-3} = \frac{5(8)+4}{2-3} = \frac{40+4}{-1} = -44$$

Theorem

consider The rational function

$$f(x) = \frac{P(x)}{q(x)}$$

where $P(x)$ and $q(x)$ are polynomial. For any real number a,

(a) if $q(a) \neq 0$, Then $\lim_{x \to a} f(x) = f(a)$.

(b) if $q(a) = 0$, but $p(a) \neq 0$ Then $\lim_{x \to a} f(x)$ does not exists.

Example:

① $\lim_{x \to 3} \frac{x^2-6x+9}{x-3} = \lim_{x \to 3} \frac{(x-3)(x-3)}{(x-3)}$

$$= \lim_{x \to 3} x-3 = 0$$

② $\lim_{x \to -4} \frac{2x+8}{x^2+x-12} = \lim_{x \to -4} \frac{2(x+4)}{(x+4)(x-3)}$

$$= \lim_{x \to -4} \frac{2}{x-3} = \frac{-2}{7}.$$

③ $\lim_{x \to 5} \frac{x^2-3x-10}{x^2-10x+25}$

$$= \lim_{x \to 5} \frac{(x-5)(x+2)}{(x-5)^2} = \lim_{x \to 5} \frac{x+2}{x-5} =$$

does not exists.

mple

Find $\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1}$ (20)

$$\Rightarrow \lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1} \cdot \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1}$$

$$= \lim_{x \to 0} \frac{x(\sqrt{x+1} + 1)}{x+1-1} = \lim_{x \to 0} \sqrt{x+1} + 1 = 2$$

Example

$$f(x) = \begin{cases} \frac{1}{x+2} & x < -2 \\ x^2 - 5 & -2 < x \leq 3 \\ \sqrt{x+13} & x > 3 \end{cases}$$

(a) $\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \frac{1}{x+2} = -\infty$

$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} x^2 - 5 = -1$

Since $\lim_{x \to -2^-} f(x) \neq \lim_{x \to -2^+} f(x)$.

$\therefore$ limit does not exist at $x = -2$

(b) $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 - 5 = 0 - 5 = -5$

(c) $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x^2 - 5 = 9 - 5 = 4$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \sqrt{x+13} = \sqrt{16} = 4$

$\therefore \lim_{x \to 3} f(x) = 4$

Example

$$\lim_{y \to 2^-} \frac{(y-1)(y-2)}{y+1} = \frac{0}{3} = 0$$

$$\lim_{x \to 3^+} \frac{x}{x-3} = +\infty .$$

computing limits (End Behavior)

Theorem let k be a real number

(1) $\lim_{x \to -\infty} k = k$ (2) $\lim_{x \to +\infty} k = k$

(3) $\lim_{x \to -\infty} x = -\infty$ (4) $\lim_{x \to +\infty} x = +\infty$

(5) $\lim_{x \to -\infty} \frac{1}{x} = 0$ (6) $\lim_{x \to +\infty} \frac{1}{x} = 0$

Example

I $\lim_{x \to -\infty} (7x^5 - 4x^3 + 2x - 9) = \lim_{x \to -\infty} 7x^5 = -\infty$

II $\lim_{x \to -\infty} (-4x^8 + 17x^3 - 5x + 1) = \lim_{x \to -\infty} -4x^8 = -\infty$

III $\lim_{x \to +\infty} \frac{3x+5}{6x-8} = \lim_{x \to +\infty} \frac{3\frac{x}{x} + \frac{5}{x}}{6\frac{x}{x} - \frac{8}{x}} = \frac{3}{6} = \frac{1}{2}$

IV $\lim_{x \to -\infty} \frac{4x^2 - x}{2x^3 - 5}$

V $\lim_{x \to -\infty} \frac{5x^3 - 2x^2 + 1}{3x + 5}$

4) $$\lim_{x\to-\infty} \frac{4x^2-x}{2x^3-5}$$

$$=\lim_{x\to-\infty} \frac{4\frac{x^2}{x^3}-\frac{x}{x^3}}{\frac{2x^3}{x^3}-\frac{5}{x^3}}$$

$$=\lim_{x\to-\infty} \frac{\frac{4}{x}-\frac{1}{x^2}}{2-\frac{5}{x^3}}$$

$$=\frac{0}{2}=0$$

5) $$\lim_{x\to-\infty} \frac{5x^3-2x^2+1}{3x+5}$$

$$=\lim_{x\to-\infty} \frac{5\frac{x^3}{x}-2\frac{x^2}{x}+\frac{1}{x}}{3\frac{x}{x}+\frac{5}{x}}$$

$$=\lim_{x\to-\infty} \frac{5x^2-2x+\frac{1}{x}}{3+\frac{5}{x}}=\infty$$

6) $$\lim_{x\to+\infty} \frac{\sqrt{x^2+2}}{3x-6}$$

Solution: as $x\to+\infty$, The values of x under consideration are positive, so we can replace $|x|$ by x. we obtain.

That is. $$\lim_{x\to+\infty} \frac{\sqrt{x^2+2}}{3x-6}=\lim_{x\to+\infty} \frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|}$$

$$= \lim_{x \to +\infty} \frac{\sqrt{x^2+2} \,/\, \sqrt{x^2}}{(3x-6)/x}$$

$$= \lim_{x \to +\infty} \frac{\sqrt{\frac{x^2}{x^2} + \frac{2}{x^2}}}{\frac{3x}{x} - \frac{6}{x}}$$

$$= \frac{\sqrt{1}}{3} = \frac{1}{3}$$

7) $\lim_{x \to -\infty} \frac{\sqrt{x^2+2}}{3x-6}$

Solution: As $x \to -\infty$, the values of x under consideration are negative, so we can replace $|x|$ by $-x$. We obtain

$$\lim_{x \to -\infty} \frac{\sqrt{x^2+2} \,/\, |x|}{(3x-6)/|x|} = \lim_{x \to -\infty} \frac{\sqrt{x^2+2} \,/\, \sqrt{x^2}}{(3x-6)/-x}$$

$$= \lim_{x \to -\infty} \frac{\sqrt{\frac{x^2}{x^2} + \frac{2}{x^2}}}{-\frac{3x}{x} + \frac{6}{x}} = \frac{-1}{3}.$$

Example: Find the following

1) $\lim_{x \to +\infty} \sqrt{x^6+5} - x^3$

$$= \lim_{x \to +\infty} \left(\sqrt{x^6+5} - x^3\right)\left(\frac{\sqrt{x^6+5} + x^3}{\sqrt{x^6+5} + x^3}\right)$$

$$= \lim_{x \to +\infty} \frac{x^6+5-(x^3)^2}{\sqrt{x^6+5} + x^3}$$

$$= \lim_{x \to +\infty} \frac{\cancel{x^6} + 5 - \cancel{x^6}}{\sqrt{x^6+5} + x^3} = \lim_{x \to +\infty} \frac{5}{\sqrt{x^6+5} + x^3}$$

$$= \lim_{x \to +\infty} \frac{\frac{5}{x^3}}{\sqrt{\frac{x^6}{x^6} + \frac{5}{x^6}} + \frac{x^3}{x^3}} = \frac{0}{1} = 0$$

② $\lim\limits_{x \to +\infty} \left(\sqrt{x^6+5x^3} - x^3\right)$

$$= \lim_{x \to +\infty} \left(\sqrt{x^6+5x^3} - x^3\right)\left(\frac{\sqrt{x^6+5x^3} + x^3}{\sqrt{x^6+5x^3} + x^3}\right)$$

$$= \lim_{x \to +\infty} \frac{\cancel{x^6} + 5x^3 - \cancel{x^6}}{\sqrt{x^6+5x^3} + x^3} = \lim_{x \to +\infty} \frac{5x^3}{\sqrt{x^6+5x^3} + x^3}$$

$$= \lim_{x \to +\infty} \frac{5\frac{x^3}{x^3}}{\sqrt{\frac{x^6}{x^6} + \frac{5x^3}{x^6}} + \frac{x^3}{x^3}}$$

$$= \lim_{x \to +\infty} \frac{5}{\sqrt{1 + \frac{5}{x^3}} + 1} = \frac{5}{2}$$

Let c be any real number in The natural domain of trigonometric function, Then

① $\lim_{x \to c} \sin x = \sin c$

② $\lim_{x \to c} \cos x = \cos c$

③ $\lim_{x \to c} \tan x = \tan c$

④ $\lim_{x \to c} \csc x = \csc c$

⑤ $\lim_{x \to c} \sec x = \sec c$

⑥ $\lim_{x \to c} \cot x = \cot c$

Remark :

(I) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(II) $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$

Now $\lim_{x \to 0} \frac{1-\cos x}{x} = \lim_{x \to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x}$

$= \lim_{x \to 0} \frac{1-\cos^2 x}{x(1+\cos x)}$

$= \lim_{x \to 0} \frac{\sin^2 x}{x(1+\cos x)}$

$= \lim_{x \to 0} \frac{\sin x}{x} \lim_{x \to 0} \frac{\sin x}{1+\cos x}$

$= (1)(0) = 0$

(3) $$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\left(\tan x\cdot\frac{1}{x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{\cos x}\cdot\frac{1}{x}\right).$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\lim_{x\to 0}\left(\frac{1}{\cos x}\right)$$

$$= (1)(1) = 1.$$

(4) $$\lim_{x\to 0}\frac{\sin 2x}{x} = \lim_{x\to 0}\frac{\sin 2x}{x}\ \frac{2}{2}$$

$$= \lim_{x\to 0}\frac{\sin 2x}{2x}\cdot 2$$

$$= 2\lim_{x\to 0}\frac{\sin 2x}{2x} = 2$$

(5) $$\lim_{x\to 0}\frac{\sin 3x}{\sin 7x}$$

$$= \lim_{x\to 0}\frac{\frac{\sin 3x}{3x}\cdot 3}{\frac{\sin 7x}{7x}\cdot 7} = \frac{3}{7}\lim_{x\to 0}\frac{\frac{\sin 3x}{3x}}{\frac{\sin 7x}{7x}}$$

$$= \frac{3}{7}\left(\lim_{x\to 0}\frac{\sin 3x}{3x} \div \lim_{x\to 0}\frac{\sin 7x}{7x}\right)$$

$$= \frac{3}{7}\left(\frac{1}{1}\right) = \frac{3}{7}$$

3. $$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\left(\tan x\cdot\frac{1}{x}\right)$$

$$=\lim_{x\to 0}\left(\frac{\sin x}{\cos x}\cdot\frac{1}{x}\right).$$

$$=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\lim_{x\to 0}\left(\frac{1}{\cos x}\right)$$

$$=(1)(1)=1.$$

4. $$\lim_{x\to 0}\frac{\sin 2x}{x}=\lim_{x\to 0}\frac{\sin 2x}{x}\;\frac{2}{2}$$

$$=\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot 2$$

$$=2\lim_{x\to 0}\frac{\sin 2x}{2x}=2$$

5. $$\lim_{x\to 0}\frac{\sin 3x}{\sin 7x}$$

$$=\lim_{x\to 0}\frac{\frac{\sin 3x}{3x}\cdot 3}{\frac{\sin 7x}{7x}\cdot 7}=\frac{3}{7}\lim_{x\to 0}\frac{\frac{\sin 3x}{3x}}{\frac{\sin 7x}{7x}}$$

$$=\frac{3}{7}\left(\lim_{x\to 0}\frac{\sin 3x}{3x}\div\lim_{x\to 0}\frac{\sin 7x}{7x}\right)$$

$$=\frac{3}{7}\left(\frac{1}{1}\right)=\frac{3}{7}$$

(6) $\lim_{\theta \to 0} \frac{\theta}{\cos\theta} = \frac{0}{\cos(0)} = \frac{0}{1} = 0$

(7) $\lim_{\theta \to 0} \frac{\theta^2}{1-\cos\theta}\left(\frac{1+\cos\theta}{1+\cos\theta}\right)$

Find $\lim_{\theta \to 0} \frac{\theta^2}{1-\cos\theta}$

$= \lim_{\theta \to 0} \frac{\theta^2(1+\cos\theta)}{1-\sin^2\theta}$

$= \lim_{\theta \to 0} \frac{\theta^2(1+\cos\theta)}{\cos^2\theta}$

$= \lim_{\theta \to 0} \frac{\theta^2}{\cos^2\theta} \lim_{\theta \to 0} (1+\cos\theta)$

$= 0(2) = 0$

(8) $\lim_{x \to 0} \frac{2x+\sin x}{x}$

$= \lim_{x \to 0} \frac{2x}{x} + \lim_{x \to 0} \frac{\sin x}{x}$

$= \lim_{x \to 0} 2 + \lim_{x \to 0} \frac{\sin x}{x}$

$= 2 + 1 = 3$

⑨ $$\lim_{x\to 5} \frac{\sin(x-5)}{x^2-25}$$

$$= \lim_{x\to 5} \frac{\sin(x-5)}{(x-5)(x+5)}$$

$$= \lim_{x\to 5} \frac{\sin(x-5)}{(x-5)} \cdot \lim_{x\to 5} \frac{1}{x+5}$$

$$= (1)\left(\frac{1}{10}\right) = \frac{1}{10}$$

More solved problems about End Behavior

① $$\lim_{x\to +\infty} (1+2x-3x^5) = \lim_{x\to +\infty} -3x^5 = -\infty$$

② $$\lim_{x\to -\infty} \frac{x-2}{x^2+2x+1} = \frac{\infty}{-\infty}$$

$$= \lim_{x\to -\infty} \frac{\frac{x}{x^2} - \frac{2}{x^2}}{\frac{x^2}{x^2} + \frac{2x}{x^2} + \frac{1}{x^2}}$$

$$= \lim_{x\to -\infty} \frac{\frac{1}{x} - \frac{2}{x^2}}{1 + \frac{2}{x} + \frac{1}{x^2}} = \frac{0-0}{1+0+0} = \frac{0}{1} = 0$$

③ $$\lim_{r\to -\infty} \frac{5-2r^3}{r^2+1} = \frac{\infty}{\infty}$$

$$\lim_{r\to -\infty} \frac{\frac{5}{r^2} - \frac{2r^3}{r^2}}{\frac{r^2}{r^2} + \frac{1}{r^2}} = \lim_{r\to -\infty} \frac{\frac{5}{r^2} - 2r}{1 + \frac{1}{r^2}} = \infty$$

(4) $\lim_{z \to -\infty} \frac{2-z}{\sqrt{7+6z^2}} = \frac{\infty}{\infty}$

$\Rightarrow \lim_{z \to -\infty} \frac{(2-z)/|z|}{\sqrt{7+6z^2}/|z|}$

$= \lim_{z \to -\infty} \frac{(2-z)/(-z)}{\sqrt{\frac{7}{z^2}+6\frac{z^2}{z^2}}}$

$= \lim_{z \to -\infty} \frac{-\frac{2}{z}+\frac{z}{z}}{\sqrt{\frac{7}{z^2}+6\frac{z^2}{z^2}}} = \lim_{z \to -\infty} \frac{-\frac{2}{z}+1}{\sqrt{\frac{7}{z^2}+6}}$

$= \frac{1}{\sqrt{6}}$

(5) $\lim_{y \to -\infty} \frac{3}{y+4} = \frac{3}{-\infty} = 0$

(6) $\lim_{h \to +\infty} \sqrt[7]{\frac{3h^7-4h^5}{2h^7+1}} = \frac{\infty}{\infty}$

$= \sqrt[7]{\lim_{h \to +\infty} \frac{3h^7-4h^5}{2h^7+1}} = \sqrt[7]{\lim_{h \to +\infty} \frac{\frac{3h^7}{h^7}-\frac{4h^5}{h^7}}{\frac{2h^7}{h^7}+\frac{1}{h^7}}}$

$= \sqrt[7]{\lim_{h \to +\infty} \frac{3-4/h^2}{2+1/h^7}} = \sqrt[7]{\frac{3}{2}}$

Definition: A function f is said to be continuous at $x=c$ provided The following conditions are satisfied

1. $f(c)$ is defined
2. $\lim_{x\to c} f(x)$ exists.
3. $\lim_{x\to c} f(x) = f(c)$.

Example: Is The function $f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ \sin x & \text{if } x < 0 \end{cases}$

(I) $f(0) = (0)^2 = 0$

(II) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sin x = \sin(0) = 0$

That is $\lim_{x\to 0} f(x) = 0$

(III) $\lim_{x\to 0} f(x) = f(0)$

$\therefore$ f is continuous

Example: check continuity of

$$g(x) = \begin{cases} \dfrac{3}{x-1} & x \neq 1 \\ 3 & x = 1. \end{cases}$$

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} \frac{3}{x-1} = \frac{3}{0} = \infty$$

$$g(1) = 3$$

Since $\lim_{x \to 1} g(x) \neq g(1)$

Hence g is not continuous at $x=1$.

...em if the function f and g are continuous at c,

then ① $f+g$ is continuous at c

② $f-g$ is continuous at c

③ $f \cdot g$ is continuous at c

④ f/g is continuous at c if $g(c) \neq 0$.

clearly f/g is defined at c it values $\frac{f(c)}{g(c)}$, $g(c) \neq 0$

Now to find $\lim_{x \to c} \frac{f(x)}{g(x)}$.

since f is continuous at c, then $\lim_{x \to c} f(x) = f(c)$

and since g is continuous at c, then $\lim_{x \to c} g(x) = g(c)$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{f(c)}{g(c)}.$$

$\therefore$ f/g is continuous at c.

Theorem if the function g is continuous at c and the function f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c.

since $\lim_{x \to c} (f \circ g)(x) = \lim_{x \to c} f(g(x)) = f(\lim_{x \to c} g(x))$

$= f(g(c))$

$= f \circ g(c)$.

Find a value for the constant K if possible, That be make the function continuous everywhere.

$$f(x) = \begin{cases} 7x-2 & x \leq 1 \\ Kx^2 & x > 1 \end{cases}$$

Derivative

: المشتقة

suppose that x_0 is number in the domain of a function f. if $\lim_{x_1 \to x_0} \frac{f(x_1)-f(x_0)}{x_1-x_0}$ exists, Then the value of This limits is called the derivative of f at $x = x_0$. and is denoted by $f'(x_0)$. That is

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x_1)-f(x_0)}{x_1-x_0}$$ is equivalent to definition of slope at $x = x_0$.

Definition suppose that x_0 is a number of in the domain of a function. if $f'(x_0) = \lim_{x_1 \to x_0} \frac{f(x_1)-f(x_0)}{x_1-x_0}$ is exists Then we can define the tangent line to the graph of ~~point~~ f at the point $(x_0, f(x_0))$

$$y - f(x_0) = f'(x_0)(x - x_0).$$

it is also call This tangent line to the graph of f at $x = x_0$.

Ex/ Find the slope of the graph of $y = x^2+1$ at the point $(2,5)$ and use it to find the equation of the tangent line to $y = x^2+1$ at $(2,5)$

$$f'(2) = \lim_{x_1 \to 2} \frac{f(x_1)-f(2)}{x_1-2} = \lim_{x_1 \to 2} \frac{x_1^2+1-5}{x_1-2}$$

$$\lim_{x_1 \to 2} \frac{(x_1-2)(x_1+2)}{(x_1-2)} = 4$$

The equation of tangent line

$$y - f(x_0) = f'(x_0)(x - x_0)$$

$$\Rightarrow y - 5 = 4(x-2)$$

$$\Rightarrow y - 5 = 4x - 8$$

$$\Rightarrow y = 4x - 3$$

Definition :- The function f' is defined by the formula

$$f'(x) = \lim_{w \to x} \frac{f(w) - f(x)}{w - x}$$

is called derivative of f w.r.t x. The domain of f' consists all x in the domain of f for which the limit exists.

Example : Find the derivative with respect to x of $f(x) = x^3 - x$

$$f'(x) = \lim_{w \to x} \frac{f(w) - f(x)}{w - x}$$

$$= \lim_{w \to x} \frac{(w^3 - w) - (x^3 - x)}{w - x}$$

$$= \lim_{w \to x} \frac{w^3 - w - x^3 + x}{w - x}$$

$$= \lim_{w \to x} \frac{(w^3 - x^3) - (w - x)}{w - x}$$

$$\lim_{w \to x} \frac{(w-x)(w^2+wx+x^2)-(w-x)}{(w-x)}$$

$$= \lim_{w \to x} \frac{\cancel{(w-x)}(w^2+wx+x^2-1)}{\cancel{(w-x)}}$$

$$= x^2+x^2+x^2-1 = 3x^2-1.$$

Example find the derivative w.r.t x of $f(x) = \sqrt{x}$

$$f'(x) = \lim_{w \to x} \frac{f(w)-f(x)}{w-x}$$

$$= \lim_{w \to x} \frac{\sqrt{w}-\sqrt{x}}{w-x} \cdot \frac{\sqrt{w}+\sqrt{x}}{\sqrt{w}+\sqrt{x}}$$

$$= \lim_{w \to x} \frac{\cancel{w-x}}{\cancel{w-x}} \cdot \frac{1}{\sqrt{w}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Derivative Notation:

① $\frac{d}{dx}[f(x)] = f'(x)$ ② $\frac{d}{dx}[f(x)]\Big|_{x_0} = f'(x_0)$.

③ $\frac{dy}{dx} = f'(x)$; $\frac{dy}{dx}\Big|_{x=x_0} = f'(x_0)$.

...tion : if $y=f(x)$ is a continuous function, then we ...define The derivative of a function as limit as

$$f'(x)=y'=\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

Example : Find The derivative of a function $y=x^2$ by definition

$$f'(x)=y'=\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+\Delta^2 x-x^2}{\Delta x}=\lim_{\Delta x\to 0}\frac{2x\Delta x+\Delta^2 x}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\Delta x(2x+\Delta x)}{\Delta x}=\lim_{\Delta x\to 0}2x+\Delta x=2x.$$

Example. Find derivative by definition for $y=\sin x$.

$$y'=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$y'=\lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin x}{\Delta x}$$

$$y'=\lim_{\Delta x\to 0}\frac{\sin x\cos\Delta x+\cos x\sin\Delta x-\sin x}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\sin x\cos\Delta x-\sin x+\cos x\sin\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x - \sin x}{\Delta x} + \lim_{\Delta x \to 0} \cos x \frac{\sin \Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x (\cos \Delta x - 1)}{\Delta x} + \cos x \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}$$

$$= \cos x .$$

Theorem

① $\frac{d}{dx}[c] = 0$.

$$y' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c - c}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 0 = 0$$

② $\frac{d}{dx}[c f(x)] =$

$$\Rightarrow y' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{c f(x+\Delta x) - c f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{c [f(x+\Delta x) - f(x)]}{\Delta x}$$

$$= c \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= c . \frac{d}{dx}[f(x)]$$

Theorem

if f and g are differentiable at x, Then $f+g$, $f-g$ are also ~~[illegible]~~ differentiable at x

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

Example

$$\frac{d}{dx}[x^5+x^3] = \frac{d}{dx}[x^5] + \frac{d}{dx}[x^3]$$

$$= 5x^4 + 3x^2.$$

Theorem * if f and g are differentiable at x, Then $f \cdot g$ is also differentiable at x.

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

*- if f and g are differentiable at x, Then $\frac{f}{g}$ is also differentiable at x.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Theorem if n is any integer, Then

$$\frac{d}{dx}[x^n] = n\,x^{n-1}$$

Higher order derivative :-

① $f'(x) = \frac{d}{dx}[f(x)]$

② $f''(x) = \frac{d}{dx}[f'(x)] = \frac{d}{dx}\left[\frac{d}{dx} f(x)\right] = \frac{d^2}{dx^2}[f(x)]$.

③ $f'''(x) = \frac{d^3}{dx^3}(f(x))$

④ $f^{(n)}(x) = \frac{d^n}{dx^n}[f(x)]$.

Examples :—

① $\frac{d}{dx}\left[\frac{1}{2}\right] = 0$

② $\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}[x^{-1}] = -x^{-1-1} = -x^{-2} = \frac{-1}{x^2}$

③ $\frac{d}{dx}\left[x^2 + \frac{3}{2}x\right] = \frac{d}{dx}[x^2] + \frac{d}{dx}\left[\frac{3}{2}x\right]$

$= \frac{d}{dx}[x^2] + \frac{3}{2}\frac{d}{dx}[x]$

$= 2x + \frac{3}{2}$.

④ $\frac{d}{dx}\left[\frac{x}{\sqrt{\pi}}\right] = \frac{1}{\sqrt{\pi}}\frac{d}{dx}[x] = \frac{1}{\sqrt{\pi}}$

⑤ $\frac{d}{dx}\left[(x^2+1)(2x+3)\right] = (x^2+1)(2) + (2x+3)(2x)$

$= 2x^2 + 2 + 4x^2 + 6x$

$= 6x^2 + 6x + 2$.

(6) $\frac{d}{dx}\left[\frac{2x}{3x+1}\right] = \frac{(3x+1)(2) - 2x(3)}{(3x+1)^2}$

$= \frac{\cancel{6x} + 2 - \cancel{6x}}{(3x+1)^2}$

$= \frac{2}{(3x+1)^2}$

(7) $\frac{d}{dx}\left[\left(\frac{2}{x} + \frac{3}{x^3}\right)(4x^2+27)\right]$

$= \frac{d}{dx}\left[(2x^{-1} + 3x^{-3})(4x^2+27)\right]$

$= (2x^{-1} + 3x^{-3})(8x) + (4x^2+27)(-2x^{-2} - (3)(3)x^{-4})$

$= \left(\frac{2}{x} + \frac{3}{x^3}\right)(8x) + (4x^2+27)\left(\frac{-2}{x^2} - 9\frac{1}{x^4}\right)$

(8) Find $\frac{d^2y}{dx^2}$ for $y = \frac{x+1}{x}$.

$y = \frac{x+1}{x} \Rightarrow y = \frac{x}{x} + \frac{1}{x}$

$\Rightarrow y = 1 + \frac{1}{x}$

Higher order derivative :-

1. $f'(x) = \frac{d}{dx}[f(x)]$
2. $f''(x) = \frac{d}{dx}[f'(x)] = \frac{d}{dx}\left[\frac{d}{dx} f(x)\right] = \frac{d^2}{dx^2}[f(x)]$.
3. $f'''(x) = \frac{d^3}{dx^3}(f(x))$
4. $f^{(n)}(x) = \frac{d^n}{dx^n}[f(x)]$.

Examples :-

1. $\frac{d}{dx}\left[\frac{1}{2}\right] = 0$

2. $\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}[x^{-1}] = -x^{-1-1} = -x^{-2} = \frac{-1}{x^2}$

3. $\frac{d}{dx}\left[x^2 + \frac{3}{2}x\right] = \frac{d}{dx}[x^2] + \frac{d}{dx}\left[\frac{3}{2}x\right]$
$= \frac{d}{dx}[x^2] + \frac{3}{2}\frac{d}{dx}[x]$
$= 2x + \frac{3}{2}$.

4. $\frac{d}{dx}\left[\frac{x}{\sqrt{\pi}}\right] = \frac{1}{\sqrt{\pi}}\frac{d}{dx}[x] = \frac{1}{\sqrt{\pi}}$

5. $\frac{d}{dx}[(x^2+1)(2x+3)] = (x^2+1)(2) + (2x+3)(2x)$
$= 2x^2 + 2 + 4x^2 + 6x$
$= 6x^2 + 6x + 2$.

(6) $\frac{d}{dx}\left[\frac{2x}{3x+1}\right] = \frac{(3x+1)(2) - 2x(3)}{(3x+1)^2}$ II

$$= \frac{\cancel{6x}+2-\cancel{6x}}{(3x+1)^2}$$

$$= \frac{2}{(3x+1)^2}$$

(7) $\frac{d}{dx}\left[\left(\frac{2}{x}+\frac{3}{x^3}\right)(4x^2+27)\right]$

$= \frac{d}{dx}\left[(2x^{-1}+3x^{-3})(4x^2+27)\right]$

$= (2x^{-1}+3x^{-3})(8x) + (4x^2+27)(-2x^{-2} - (3)(3)x^{-4})$

$= \left(\frac{2}{x}+\frac{3}{x^3}\right)(8x) + (4x^2+27)\left(\frac{-2}{x^2} - 9\frac{1}{x^4}\right)$

(8) Find $\frac{d^2y}{dx^2}$ for $y = \frac{x+1}{x}$.

$y = \frac{x+1}{x} \Rightarrow y = \frac{x}{x} + \frac{1}{x}$

$\Rightarrow y = 1 + \frac{1}{x} \Rightarrow y = 1 + x^{-1}$

$\frac{dy}{dx} = -x^{-1-1} = -x^{-2}$

$= \frac{-1}{x^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{-1}{x^2}\right) = \frac{d}{dx}\left(-x^{-2}\right)$

$= -(-2)x^{-2-1}$

$= 2x^{-3}$

III

$$\frac{d^2y}{dx^2} = \frac{2}{x^3}.$$

Derivatives of Trigonometric Functions:-

① $\frac{d}{dx}[\sin x] = \cos x$

② $\frac{d}{dx}[\cos x] = -\sin x$

③ $\frac{d}{dx}[\tan x] = \sec^2 x.$

④ $\frac{d}{dx}[\sec x] = \sec x \tan x.$

⑤ $\frac{d}{dx}[\cot x] = -\csc^2 x$

⑥ $\frac{d}{dx}[\csc x] = -\csc x \cot x.$

Example: Find $f'(x)$ if $f(x) = x^3 \cdot \tan x$

solution: $f'(x) = x^3 \cdot \sec^2 x + \tan x \,(3x^2).$

Example: Find $f'(x)$ if ① $f(x) = \frac{\cos x}{x \sin x}$

First method

$$f(x) = \frac{\cos x}{x \sin x} = \frac{1}{x}\frac{\cos x}{\sin x} = \frac{1}{x}\cot x.$$

$$f'(x) = \frac{1}{x}(-\csc^2 x) + \cot x\left(-\frac{1}{x^2}\right)$$

$$= \frac{-1}{x}\csc^2 x - \frac{1}{x^2}\cot x.$$

Second method

(IV)

$$f(x) = \frac{\cos x}{x \sin x}$$

$$f'(x) = \frac{x \sin x(-\sin x) - \cos x\,(x \cos x + \sin x)}{(x \sin x)^2}$$

$$= \frac{-x \sin^2 x - x \cos^2 x - \cos x \sin x}{(x \sin x)^2}$$

$$= \frac{-x(\sin^2 x + \cos^2 x) - \cos x \sin x}{(x \sin x)^2}$$

$$= \frac{-x - \cos x \sin x}{(x \sin x)^2}$$

$$= \frac{-x - \cos x \sin x}{x^2 \sin^2 x}$$

$$= \frac{-1}{x} \csc^2 x - \frac{1}{x^2} \cot x.$$

(2) $y = \sin^2 x + \cos^2 x$

$$\frac{dy}{dx} = 2 \sin x \cos x + 2 \cos x (-\sin x)$$

$$= 2 \sin x \cos x - 2 \sin x \cos x = 0$$

or simply

$$y = \sin^2 x + \cos^2 x$$

$$= 1$$

$$\frac{dy}{dx} = 0$$

3. $$f(x) = \frac{\cot x}{1+\csc x}$$

$$f'(x) = \frac{(1+\csc x)(-\csc^2 x) - \cot x(-\csc x \cdot \cot x)}{(1+\csc x)^2}$$

$$= \frac{-\csc^2 x(1+\csc x) + \cot^2 x \csc x}{(1+\csc x)^2}$$

4. $$f(x) = \frac{\sec x}{1+\tan x}$$

$$f'(x) = \frac{(1+\tan x)\sec x \tan x - \sec x \cdot \sec^2 x}{(1+\tan x)^2}$$

5. Show That $y = A\sin x + B\cos x$ is solution of The equation $y'' + y = 0$

$y = A\sin x + B\cos x$

$y' = A\cos x - B\sin x$

$y'' = -A\sin x - B\cos x$

Then $y'' + y = -A\sin x - B\cos x + A\sin x + B\cos x$

$= 0$

Chain rule:

Theorem: if g is differentiable at x and f is differentiable at $g(x)$, Then The composition $f \circ g$ is differentiable at x

$$(f \circ g)'(x) = f'(g(x))\, g'(x).$$

Alternatively, if $y = f(g(x))$ and $U = g(x)$, Then $y = f(U)$. and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example :- Find $h'(x)$ if $h(x) = 3\sec(x^4)$

Solution:

Let $U = x^4$ $\Rightarrow \frac{du}{dx} = 4x^3$

$y = h(u) = 3\sec(U) \Rightarrow \frac{dy}{du} = 3\sec(U)\tan(U)$.

$$h'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 3\sec(U)\tan(U)(4x^3).$$

$$= 3\sec(x^4)\tan(x^4)(4x^3)$$

$$= 12x^3 \sec(x^4)\tan(x^4).$$

Example: Find $\frac{dw}{dt}$ if $w = \cos x$ $x = 3t^2+1$

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}$$

$\frac{dw}{dx} = -\sin x$; $\frac{dx}{dt} = 6t$

$$\Rightarrow \frac{dw}{dt} = (-\sin x)(6t).$$

$$= -\sin(3t^2+1)(6t) = -6t\sin(3t^2+1).$$

~~Ro~~lle's Theorem ①

Let f be differentiable on (a,b) and continuous on $[a,b]$. if $f(a)=f(b)=0$ Then There is at least one number c in (a,b) such That $f'(c)=0$.

لتحقيق نظرية رول يجب ان نتأكد من تحقق
شروطها ① f قابلة للاشتقاق في فترة
المفتوحة (a,b) ② f مستمرة عند الفترة
المغلقة $[a,b]$ ③ ايجاد قيم $f(b)$ و $f(a)$
بحيث $f(a)=f(b)=0$ عند ذلك تكون دالة
محققة لنظرية رول.

Example verify The hypothesis of Rolle's Th and conclusion. for

$f(x)=x^2-6x+8$ in The interval $[2,4]$

① f is diff. on $(2,4)$ ② f is continuous on $[2,4]$

③ $f(2)=4-6(2)+8=0$

$f(4)=16-8(4)+8=16-24+8=0$.

نلاحظ ان الدالة تحقق الشروط الثلاثة
الآن يمكن ايجاد قيمة c

ple

erify the hypothesis and conclusion of Rolle's Theorem on $f(x)=x^2+2x$ in the interval $[-2,0]$

ans/

1) f is continuous in $[-2,0]$

2) f is differentiable on $(-2,0)$

3) $f(-2)=(-2)^2+2(-2)=4-4=0$

$f(0)=(0)^2+2(0)=0$

so it is satisfied the hypothesis of Rolle's Th.

Now, Find c

$f(x)=x^2+2x \Rightarrow f'(x)=2x+2$

$\Rightarrow f'(c)=2c+2=0$

$\Rightarrow 2c+2=0 \Rightarrow 2c=-2$

$\Rightarrow c=-1$

$f(x) = x^3 - 3x^2 + 2x$

$f(0) = 0$

$f(2) = 8 - 3(4) + 4$

$= 0$

H.W (Rolle's Theorem)

① $f(x) = x^3 - 3x^2 + 2x \quad [0, 2]$

② $f'(x) = \dfrac{(x-2)(2x) - (x^2-1)}{(x-2)^2}$

$f'(c) = \dfrac{2x^2 - 4x - x^2 + 1}{(x-2)}$

$f'(c) = \dfrac{\cancel{2}c^2 - 4c + 1}{c-2} = \dfrac{(c-\cancel{2})(}{}$

ample: verify That hypotheses of Rollé's Th are satisfied on The interval $[0,4]$ for $f(x) = \frac{1}{2}x - \sqrt{x}$

Solution:

لتحقق من الشروط الثلاثة لنظرية رول. لاحظ ان الدالة تحتوي على $\sqrt{x}$ لكن ليس لمشكلة لأنه الفترة المعطاة في السؤال هي $[0,4]$ والتي تكون فيه الجذر $\sqrt{x}$ عدد حقيقي وبالتالي تكون دالة مستمرة وقابلة للاشتقاق.

① f is differentiable on $(a,b) = (0,4)$

② f is continuous on $[a,b] = [0,4]$.

③ $f(a) = f(0) = 1/2\,(0) - \sqrt{0} = 0$

$f(b) = f(4) = 1/2\,(4) - \sqrt{4} = 2 - 2 = 0$

اذن يمكن تطبيق نظرية رول وايجاد قيم c

$$f(x) = \frac{1}{2}x - \sqrt{x}$$

$$\Rightarrow f'(x) = \frac{1}{2} - \frac{1}{2\sqrt{x}}$$

$$\Rightarrow f'(c) = \frac{1}{2} - \frac{1}{2\sqrt{c}}$$

set $f'(c) = 0 \Rightarrow \frac{1}{2} - \frac{1}{2\sqrt{c}} = 0$

(4)

...f be differentiable on (a,b) and continuous on $[a,b]$, Then There is at least one number c in (a,b) such That $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Example (1) show That satisfies The hypothesis of Mean Value Theorem on The interval $[0,2]$ and find all values of c in $(0,2)$ which is guaranteed by The mean value Theorem for $f(x) = \dfrac{x^3}{4} + 1$

ans: لاحظ ان الدالة f مستمرة وقابلة للاشتقاق
لأن $f(x) = \dfrac{x^3}{4} + 1$ لا يمكن ان يكون مقامها صفر
$4 \neq 0$
كذلك لا تحتوي على جذر.

ans/ f is differentiable on $(a,b) = (0,2)$
and f is continuous at $[a,b] = [0,2]$

Now, since $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$f(b) = f(2) = \dfrac{(2)^3}{4} + 1 = \dfrac{8}{4} + 1 = 3$

$f(a) = f(0) = \dfrac{(0)^3}{4} + 1 = 0 + 1 = 1$

الآن عرفنا قيمة $f(b)$ و $f(a)$ وكذلك a , b معروفة
يتبقى الآن ان نجد قيمة $f'(c)$

يمكن الحصول على قيمة $f'(c)$ من خلال اشتقاق الدالة f ثم نعوض محل x بـ c كما يلي

Since $f(x) = \frac{x^3}{4} + 1$

Then $f'(x) = \frac{3}{4}x^2 \Rightarrow f'(c) = \frac{3}{4}c^2$

$\because f'(c) = \frac{f(b) - f(a)}{b - a}$

$$\Rightarrow \frac{3}{4}c^2 = \frac{3-1}{2-0} \Rightarrow \frac{3}{4}c^2 = \frac{2}{2}$$

$$\Rightarrow \frac{3}{4}c^2 = 1$$

$$\Rightarrow 3c^2 = 4$$

$$\Rightarrow c^2 = \frac{4}{3} \Rightarrow c = \mp\sqrt{\frac{4}{3}}$$

Clearly, the only value of $c \in (0,2)$ is $\sqrt{\frac{4}{3}}$

القيمة الوحيدة لـ c التي تكون موجودة في الفترة المفتوحة $(0,2)$ هي $\sqrt{\frac{4}{3}}$

Example (2) show that f satisfies the hypothesis of the mean value theorem on in the interval $[-4,6]$ and find all values of c in $(-4,6)$ which satisfy the conclusion of the theorem for $f(x) = x^2 + x$

tion :- f is differentiable on $(a,b) = (-4,6)$

and f is continuous on $[a,b] = [-4,6]$

Now, since $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ — ①

علينا ايجاد $f(b)$ و $f(a)$ وكذلك $f'(c)$

$f(a) = f(-4) = (-4)^2 + (-4) = 16 - 4 = 12$

$f(b) = f(6) = 36 + 6 = 42$

since $f(x) = x^2 + x$

Then $f'(x) = 2x + 1$

$\Rightarrow f'(c) = 2c + 1$

subtitute in ①

$$2c + 1 = \frac{42-12}{6-(-4)}$$

$$\Rightarrow 2c + 1 = \frac{30}{10}$$

$\therefore 2c + 1 = 3$

$2c = 3 - 1$

$2c = 2$

$c = \frac{2}{2} = 1$

$\in [-4,6]$

$f(x) = x^3 + x - 4$ on $[-1, 2]$

clearly f is differentiable on $(-1,2)$

and f is continuous on $[-1,2]$

Then $f(b) = f(2) = (2)^3 + 2 - 4$

$= 8 + 2 - 4 = 10 - 6 = 4$

$f(a) = f(-1) = (-1)^3 + (-1) - 4$

$= -1 - 1 - 4 = -6$

$f(x) = x^3 + x - 4$

$f'(x) = 3x^2 + 1$

$f'(c) = 3c^2 + 1$

$$3c^2 + 1 = \frac{4-(-6)}{2-(-1)} = \frac{10}{3}$$

$$3c^2 = \frac{10}{3} - \frac{3}{3} = \frac{7}{3}$$

$$\Rightarrow c^2 = \frac{7}{9} \Rightarrow c = \mp\sqrt{\frac{7}{9}}$$

Chain rule   More details (Step by Step.)

Example :- By using Chain rule Find the derivative of $h(x) = \sqrt{2x+3}$

① write $h(x)$ as composition of two functions.

Let $f(x) = \sqrt{x}$ and $g(x) = 2x+3$

That is. $h(x) = f \circ g(x)$

As seen in the following.

$f \circ g(x) = f(2x+3) = \sqrt{2x+3} = h(x).$

② $h = f(g(x))$ and let $u = g(x)$.

$\Rightarrow h = f(u)$

$$\frac{dh}{dx} = \frac{dh}{du} \cdot \frac{du}{dx}$$

Since $h = f(g(x))$
$= f(u)$
$= \sqrt{u}$

$$\Rightarrow \frac{dh}{du} = \frac{1}{2\sqrt{u}}$$

Similarly, $u = g(x)$
$= 2x+3$

$$\frac{du}{dx} = 2.$$

$$\frac{dh}{dx} = \frac{1}{2\sqrt{u}} \cdot 2 = \frac{1}{\sqrt{u}}$$

$$= \frac{1}{\sqrt{2x+3}}$$

Example (2)

By using chain rule . find the derivative of.

$$y = \frac{1}{(2x^3-7)^4}$$

I). write y as composition of two functions.

Let $f(x) = \frac{1}{x^4}$ and $g(x) = 2x^3 - 7$

Then $y = f \circ g(x)$.

II). $y = f(g(x))$ and let $u = g(x)$.

$\Rightarrow y = f(u)$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Since $y = f(u)$, then $y = \frac{1}{u^4}$

$\Rightarrow y = u^{-4}$

$$\frac{dy}{du} = -4u^{-4-1} = -4u^{-5} = \frac{-4}{u^5}$$

Since $u = g(x)$, then $u = 2x^3 - 7$.

$$\frac{du}{dx} = 6x^2.$$

Then $\frac{dy}{dx} = \frac{-4}{u^5} \cdot 6x^2$

$$= \frac{-4}{(2x^3-7)^5} \cdot 6x^2 = \frac{-24x^2}{(2x^3-7)^5}$$

Example (3)

by using chain rule. find the derivative of.

$y = \sin(x^2+3)$

1) Write y as composition of two functions

- Let $f(x) = \sin x$ $\quad g(x) = x^2+3$

Then $y = f \circ g(x)$ [check].

2). $y = f(g(x))$ and let $u = g(x)$.

$\Rightarrow y = f(u)$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Since $y = f(u)$, Then $y = \sin u$.

$$\frac{dy}{du} = +\cos u.$$

and Since $u = g(x) = x^2+3$

$$\frac{du}{dx} = 2x.$$

That is. $\dfrac{dy}{dx} = \cos u \cdot (2x)$

$= \cos(x^2+3)\ (2x).$

Example (4) find $\frac{dh}{dt}$ if $h = k^4 + 8$

$k = \tan(3t)$.

Solution $\frac{dh}{dt} = \frac{dh}{dk} \cdot \frac{dk}{dt}$

$=$

Since $h = k^4 + 8 \Rightarrow \frac{dh}{dk} = 4k^3$

and since $k = \tan(3t) \Rightarrow \frac{dk}{dt} = \sec^2(3t)(3)$

Therefore $\frac{dh}{dt} = 4k^3 \sec^2(3t)(3)$

$= 12k^3 \sec^2(3t)$

$= 12 \tan^3(3t) \sec^2(3t)$.

Rollé's Theorem:

Let f be differentiable on (a,b) and continuous on $[a,b]$ if $f(a) = f(b) = 0$, Then There is at least one number c in (a,b) such That $f'(c) = 0$

Example: Verify The hypothesis of Rollé's Theorem and conclude for $f(x) = x^2 - 6x + 8$ in the interval $[2,4]$.

① clearly f is differentiable on $(2,4)$.

② and f is continuous on $[2,4]$

③ $f(2) = 4 - 6(2) + 8 = 0$

$f(4) = 16 - 6(4) + 8 = 0$

The Three conditions are satisfied.

Since $f(x) = x^2 - 6x + 8$, Then $f'(x) = 2x - 6$

set $f'(c) = 2c - 6 = 0$

$\Rightarrow 2c = 6 \Rightarrow c = \frac{6}{2} = 3$

Example ②: check The validity of Rollé's Theorem for the function $f(x) = \frac{x+3}{x-4}$ on the interval $[1,3]$

solution: note That f is differentiable on $(1,3)$. and f is continuous on $[1,3]$

But $f(1) = \frac{1+3}{1-4} = \frac{4}{-3} \neq 0$. so we can not apply Rollés Theorem.

Example 3: verify the hypotlesis and conclusion of Rollés Theorem on $f(x) = x^2 + 2x$ on The interval $[-2,0]$.

answer: Clearly ① f is differentiable on $(-2,0)$

and ② f is continuous in $[-2,0]$.

③ $f(-2) = (-2)^2 + 2(-2) = 4 - 4 = 0$

$f(0) = (0)^2 + 2(0) = 0 + 0 = 0$

so, it is satisfied the hypothesis of Rollé's Theorem

To find the value of c.

since $f(x) = x^2 + 2x$, Then $f'(x) = 2x + 2$.

$f'(c) = 2c + 2 = 0$

$\Rightarrow c = -1 \in (-2,0)$

Example: verify the hypothesis of Rolle's Theorem are satisfied on the interval $[0,4]$ and conclusion for $f(x) = \frac{1}{2}x - \sqrt{x}$. (Home-work).

Mean Value Theorem

Let f be differentiable on (a,b) and continuous $[a,b]$, then there is at least one number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Example: Show that f satisfies the hypothesis of the mean value Theorem on the interval $[0,2]$ and find all values of c in $(0,2)$ which is obtained by the mean value Theorem for $f(x) = \frac{x^3}{4} + 1$.

answer: f is differentiable on $(0,2)$. and

f is continuous on $[0,2]$.

so, it is satisfied the hypothesis of mean value Theorem

Now, since $f'(c) = \frac{f(b)-f(a)}{b-a}$

and since $f(2) = \frac{(2)^3}{4} + 1 = \frac{8}{4} + 1 = 3$

$f(0) = \frac{0}{4} + 1 = 0 + 1 = 1$.

Now, by taking derivative of $f(x) = \frac{x^3}{4} + 1$

$f'(x) = \frac{3}{4}x^2 \Rightarrow f'(c) = \frac{3}{4}c^2$

$\Rightarrow \frac{3}{4}c^2 = \frac{3-1}{2-0} \Rightarrow \frac{3}{4}c^2 = 1$

$$c^2 = \frac{4}{3} \Rightarrow c = \mp\sqrt{\frac{4}{3}}$$

That is $c = \sqrt{\frac{4}{3}} \in (0,2)$.

Example (2) : Show that f satisfies the hypothesis of the mean value theorem in the interval $[-4,6]$ and find all values of c in $(-4,6)$ which satisfy the conclusion of the theorem for $f(x) = x^2 + x$.

<u>solution</u> f is differentiable on $(-4,6)$ and f is continuous in $[-4,6]$.

Now, we have to find $f(a) = f(-4)$
$f(b) = f(6)$

$$f(-4) = (-4)^2 + (-4) = 16 - 4 = 12$$

$$f(6) = (6)^2 + 6 = 36 + 6 = 42$$

Since $f(x) = x^2 + x$

$$f'(x) = 2x + 1.$$

$$\Rightarrow f'(c) = 2c + 1$$

Since $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Then $2c + 1 = \dfrac{42 - 12}{6 - (-4)}$.

$$2c + 1 = \frac{30}{10} \quad \Rightarrow c = 1 \in [-4,6].$$

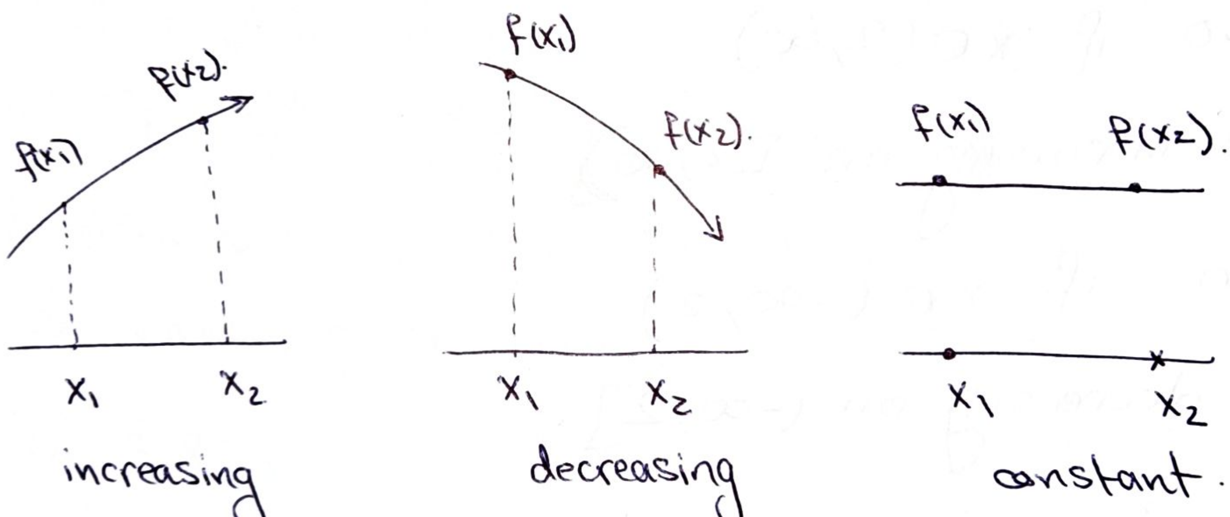
Definition

Let f be defined on an interval, and let x_1 and x_2 denote numbers in that interval.

① f is increasing on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

② f is decreasing on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

③ f is constant on the interval if $f(x_1) = f(x_2)$ for all x_1 and x_2.



Theorem Let f be a continuous function on closed interval $[a,b]$ and differentiable on the open interval (a,b), then

(1) if $f'(x) > 0$ for every value of x in (a,b), then f is increasing on $[a,b]$.

(2) if $f'(x) < 0$ for every value of x in (a,b), then f is decreasing on $[a,b]$.

(3) if $f'(x) = 0$ for every value of x in (a,b), then f is constant on $[a,b]$.

Example

Find the intervals on which the following functions are increasing and the intervals on which they are decreasing

① $f(x) = x^2 - 4x + 3$

Then $f'(x) = 2x - 4$
$= 2(x-2)$.

$- \; - \; - \; - \;\; \underset{2}{\bullet} \;\; + \; + \; + \; +$

$f'(x) > 0$ if $x \in (2, \infty)$

$\Rightarrow$ f is increasing on $[2, \infty)$

$f'(x) < 0$ if $x \in (-\infty, 2)$.

$\Rightarrow$ f is decreasing on $(-\infty, 2]$.

② $f(x) = x^3 + 8$

Then $f'(x) = 3x^2$.

$+ \; + \; + \; + \; + \;\; \underset{0}{\bullet} \;\; + \; + \; + \; + \; +$

$f'(x) > 0$ if $(-\infty, 0) \cup (0, \infty)$.

$\Rightarrow$ f is increasing on $(-\infty, 0]$
f is increasing on $[0, \infty)$.

$\Rightarrow$ f is increasing on $\mathbb{R}$.

Theorem

Let f be twice differentiable on an open interval I.

(1) if $f''(x) > 0$ on I, Then f is concave up on I

(2) If $f''(x) < 0$ on I, Then f is concave down on I.

Example :- Find open intervals on which the following functions are concave up and open interval on which They are concave down.

(1) $f(x) = x^2 + 5x + 9$

Then $f'(x) = 2x$, and $f''(x) = 2$.

Since $f''(x) = 2 > 0$ for each x, Then f is concave up on $(-\infty, \infty)$.

(b) $f(x) = 2x^3$

Then $f'(x) = 6x^2$ and $f''(x) = 12x$.

$$\underset{}{- - - -} \quad \underset{0}{\cdot} \quad + + + +$$

$f''(x) > 0$ if $x \in (0, \infty)$. $\Rightarrow$ f is concave up on $(0, \infty)$.

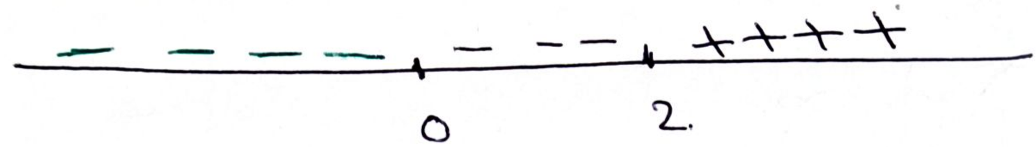
$f''(x) < 0$ if $x \in (-\infty, 0)$ $\Rightarrow$ f is concave down on $(-\infty, 0)$

Example

③ $f(x) = x^3 - 3x^2 + 1$.

Then $f'(x) = 3x^2 - 6x$.

$= 3x(x-2)$.



1) $f'(x) > 0$ if $x \in (2, \infty)$

Then f is increasing on $[2, \infty)$.

2) $f'(x) < 0$ if $x \in (0, 2)$

Then f is decreasing on $[0, 2]$

also, $f'(x) < 0$ if $x \in (-\infty, 0)$

Then f is decreasing on $(-\infty, 0]$.

$\Rightarrow$ f is decreasing on $(-\infty, 2]$.

Definition. if f is differentiable on open interval I, Then f is said to be concave up on I if f' is increasing on I

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and f is said to be concave down on I if f' is decreasing on I.

2) $f(x) = x^3 - 3x^2 + 1.$

$\Rightarrow f'(x) = 3x^2 - 6x.$

$f''(x) = 6x - 6$

$- - - - - - - - \quad + + + + +$

$\qquad\qquad 1$

$f''(x) > 0$ if $x \in (1, \infty)$. Then f is concave up on $(1, \infty)$.

$f''(x) < 0$ if $x \in (-\infty, 1)$, Then f is concave down on $(-\infty, 1)$.

Definition: if f is continuous on an open interval containing a value x_0 and if f changes concavity at
the direction
point $(x_0, f(x_0))$., Then we say That f has an inflection point at x_0.

Example :- Find the inflection points of $f(x) = \sin x$ on $[0, 2\pi]$ and confirm That your results are ~~satisfied~~ consistent With the graph of The function.

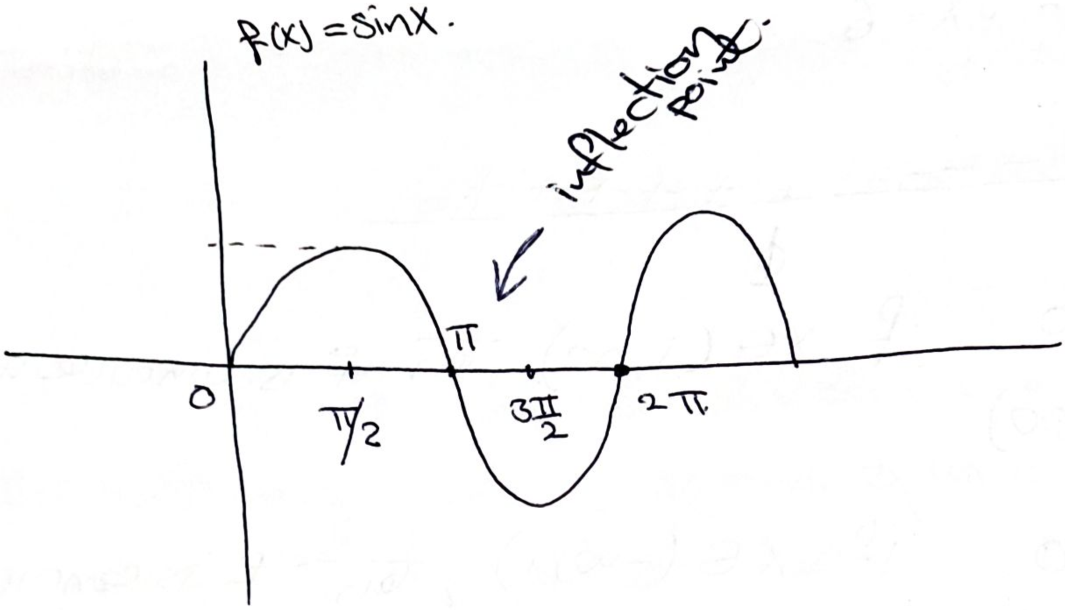
solution: $f(x) = \sin x \Rightarrow f'(x) = \cos x.$

$\Rightarrow f''(x) = -\sin x.$

$\therefore f''(x) < 0$ if $0 < x < \pi.$

$f''(x) > 0$ if $\pi < x < 2\pi$

That implies. That graph is concave down for $0 < x < \pi$ and concave up for $\pi < x < 2\pi$.



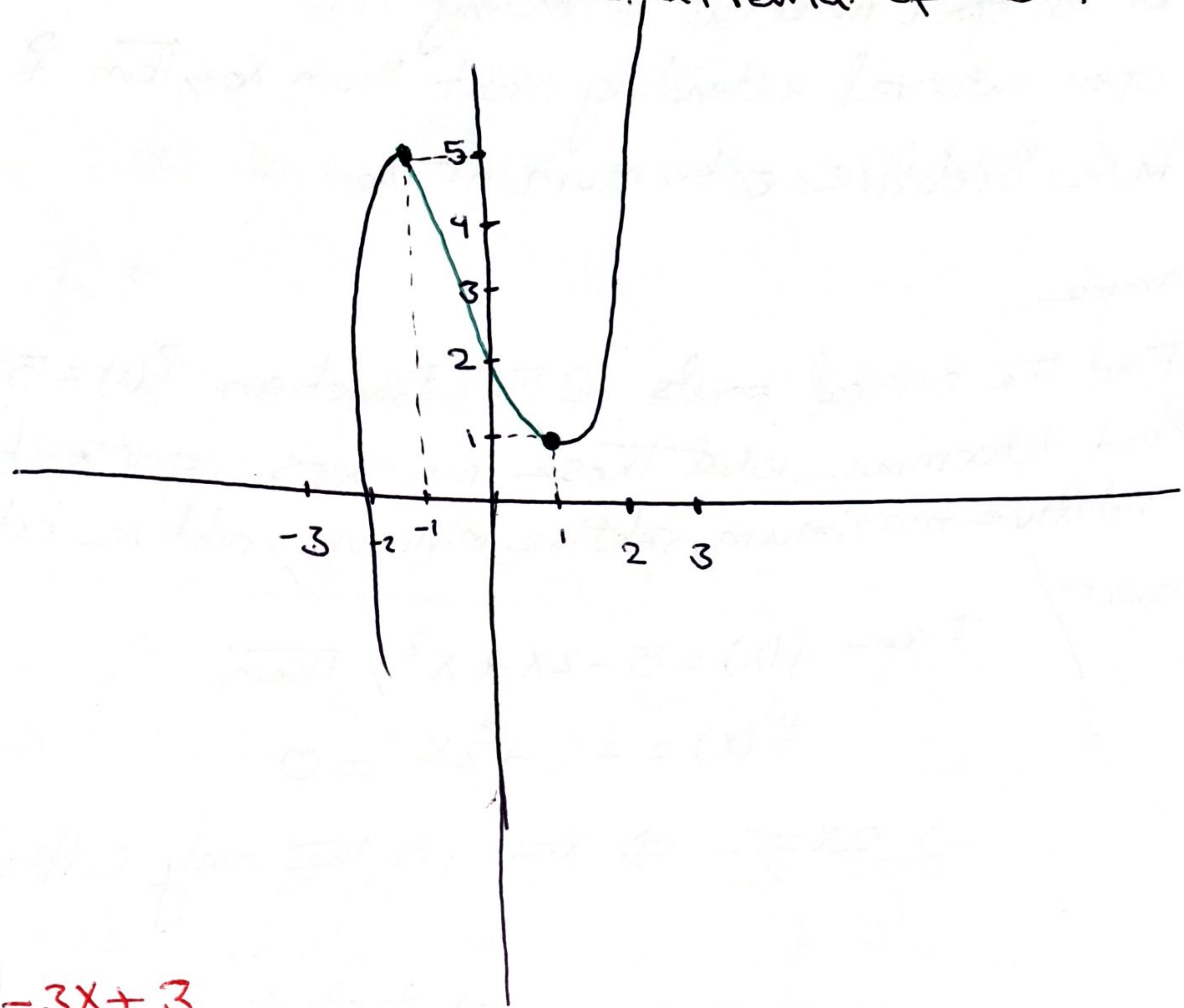
Definition: A function f is said to have a relative maximum at x_0 if There is an open interval containing x_0 on which $f(x_0)$ is the largest value, $f(x_0) \geq f(x)$ for all x in the interval.

Similarly, f is said to have relative minimum at x_0, if there is an open interval containing x_0 on which ~~is~~ $f(x_0)$ is The smallest value, $f(x_0) \leq f(x)$ for all x in The interval

if f has either relative maximum or relative minimum at x_0, Then f is said to have a relative extremum at x_0.

Definition: Values in the domain of f at which either $f'(x) = 0$ or f is not differentiable are called critical point.

Example : locate The relative extrema of This function.



$Y = x^3 - 3x + 3$

This function has relative maximum at $x = -1$ and relative minimum at $x = 1$.

Theorem (First Derivative Test.)

Suppose f is continuous at a critical number x_0.

① If $f'(x) > 0$ on an open interval extending left from x_0. and $f'(x) < 0$ on an open interval right from x_0, Then f has relative maximum at x_0.

②) If $f'(x) < 0$ an open interval extending left from x_0. and $f'(x) > 0$ on open interval extending right from Then f has relative minimum at x_0.

③ if $f'(x)$ has the same sign [either $f'(x) > 0$ o
on an open interval extending left from x_0 and on
open interval extending right from x_0, then f does n
have a relative extremum at x_0.

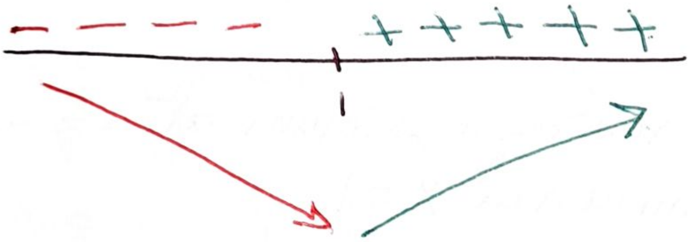
Example

Find the critical points of the function $f(x) = 5 - 2x + x^2$ and determine what those numbers represented. relative maximum, relative minimum, relative extremum.

answer/ Since $f(x) = 5 - 2x + x^2$, then

$$f'(x) = -2 + 2x = 0$$

$\Rightarrow 2x = 2 \Rightarrow x = 1$ is the only critical point.



Then ~~that~~ There is relative minimum at $x = 1$.

Example :- locate the relative maximum, relative minimum, and relative extremum of $f(x) = 3x^{5/3} - 15x^{2/3}$ if any.

answer :- The function f is defined and continuous for all real values of x.

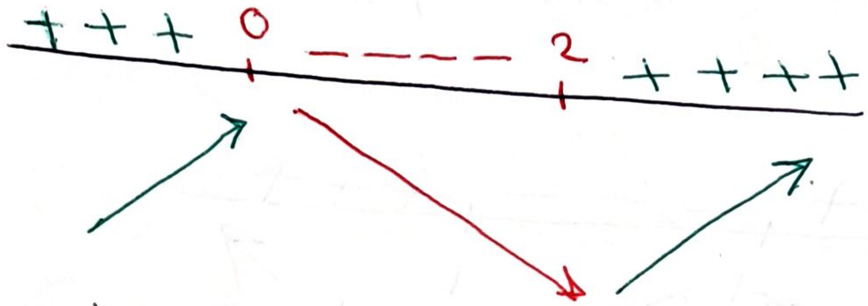
$$f'(x) = 5x^{2/3} - 10x^{-1/3}$$

$$= 5x^{-1/3}\left(\frac{x^{2/3}}{x^{-1/3}} - 2\frac{x^{-1/3}}{x^{-1/3}}\right)$$

$= 5x^{-1/3}(x-2) = \dfrac{5(x-2)}{x^{1/3}}$

Since $f'(x)$ does not exists at $x=0$ and $f'(x)=0$ if $x=2$.

So, There are two critical points $x=0$, $x=2$



so we have relative maximum at $x=0$.
and relative minimum at $x=2$

Example: utilize the given derivative to find all critical points of f

① $f'(x) = x^3(x^2-9)$.

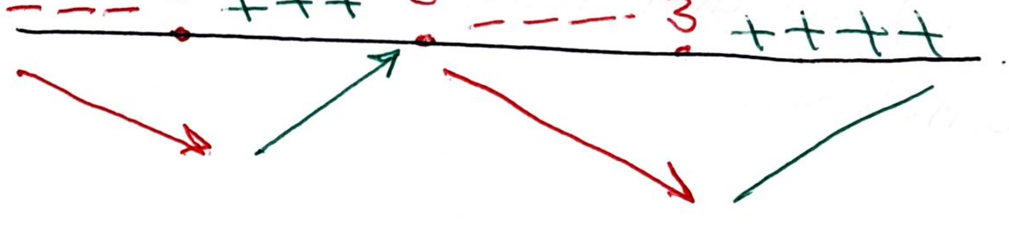
② $f'(x) = \dfrac{x^2-1}{x^2+1}$.

solution:

① $f'(x) = x^3(x^2-9) = 0$

$\Rightarrow \quad x^3(x-3)(x+3) = 0$.

so, The critical points are $x = 0, 3, -3$

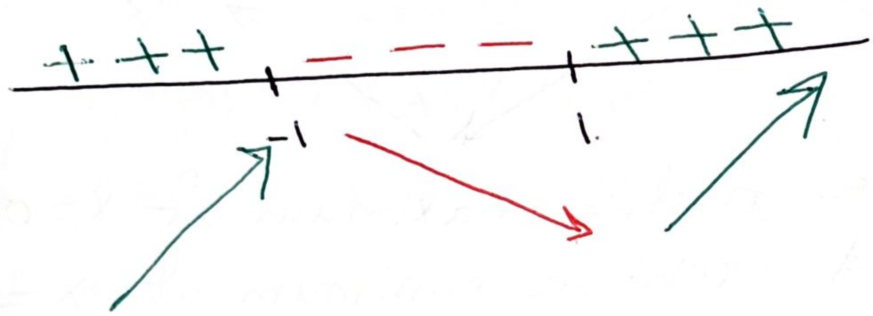


So, we have relative minimum at $-3, 3$
and relative maximum at 0

② $f'(x) = \dfrac{x^2-1}{x^2+1}$

Then $f'(x) = \dfrac{x^2-1}{x^2+1} = \dfrac{(x-1)(x+1)}{x^2+1} = 0$

Therefore the critical points. $x = -1, 1$.



So, we have the relative maximum at $x = -1$
and the relative minimum at $x = 1$.

Theorem (Second Derivative Theorem).

Suppose that f is twice differentiable at x_0

① if $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a relative minimum at x_0.

② if $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a relative maximum at x_0.

③ if $f'(x_0) = 0$ and $f''(x_0) = 0$, then the Test is inconclusive.

... the relative maxima and minima of $f(x) = x^4 - 2x^2$

solution: $f'(x) = 4x^3 - 4x$

$$= 4x(x^2-1) = 4x(x-1)(x+1). \quad \text{---} (*)$$

$$\Rightarrow f''(x) = 12x^2 - 4.$$

From (*), we have the following critical points.
$x = 0, -1, 1$.

$f''(0) = -4 < 0 \Rightarrow$ relative maximum at $x=0$
$f''(1) = 8 > 0 \Rightarrow$ relative minimum at $x=1$.
$f''(-1) = 8 > 0 \Rightarrow$ relative minimum at $x=-1$